Contextual Conditional Reasoning

Giovanni Casini[1,2] **Thomas Meyer**[2,1] **Ivan Varzinczak**[3,4,1]
[1] ISTI–CNR, Italy
[2] CAIR and University of Cape Town, South Africa
[3] CRIL, Univ. Artois & CNRS, France
[4] CAIR, Computer Science Division, Stellenbosch University, South Africa
giovanni.casini@isti.cnr.it, tmeyer@cs.uct.ac.za, varzinczak@cril.fr

Abstract

We extend the expressivity of classical conditional reasoning by introducing *context* as a new parameter. The enriched conditional logic generalises the defeasible conditional setting in the style of Kraus, Lehmann, and Magidor, and allows for a refined semantics that is able to distinguish, for example, between *expectations* and *counterfactuals*. In this paper we introduce the language for the enriched logic and define an appropriate semantic framework for it. We analyse which properties generally associated with conditional reasoning are still satisfied by the new semantic framework, provide a suitable representation result, and define an entailment relation based on Lehmann and Magidor's generally-accepted notion of Rational Closure.

1 Introduction

Conditionals are at the heart of human everyday reasoning and play an important role in the logical formalisation of reasoning. They can usually be interpreted in many ways: necessity, presumption, deontic, causal, probabilistic, counterfactual, and many others. Two very common interpretations, that are also strongly interconnected, are conditionals representing *expectations* ('If it is a bird, then presumably it flies'), and conditionals representing *counterfactuals* ('If Napoleon had won at Waterloo, all Europe would be speaking French'). Although they are connected by virtue of being conditionals, the types of reasoning they aim to model differ somewhat. E.g., the first example above assumes that the premises of conditionals are consistent with what is believed, while the second example assumes that those premises are inconsistent with an agent's beliefs. That this is problematic can be made concrete with an extended version of the (admittedly over-used) penguin example.

Example 1. *Suppose we know that birds usually fly, that penguins are birds that usually do not fly, that dodos were birds that usually did not fly, and that dodos do not exist anymore. As outlined in more detail in Example 2, the standard preferential semantic approach to representing conditionals (Lehmann and Magidor 1992) is limited in that it allows for two forms of representation of an agent's beliefs. In the one, it would be impossible to distinguish between atypical (exceptional) entities such as penguins, and non-existing entities such as dodos. In the other, it would be possible to draw this type of distinction, but at the expense of being unable to reason coherently about counterfactuals—the agent would be forced to conclude anything and everything from the existence of dodos.*

In this work we introduce a logic of *contextual* conditionals to overcome this problem. The central insight is that adding an explicit notion of context to standard conditionals allows for a refined semantics of this enriched language in which the problems described in Example 1 can be dealt with adequately. It also allows us to reason coherently with counterfactual conditionals such as 'Had Mauritius not been colonised, the dodo would not fly'. Moreover, it is possible to reason coherently with contextual conditionals without needing to know whether their premises are plausible or counterfactual. In the case of penguins and dodos, for example, it allows us to state that penguins usually fly in the context of penguins existing, and that dodos usually fly in the context of dodos existing, while being unaware of whether or not penguins and dodos actually exist. At the same time, it remains possible to make statements about what necessarily holds, regardless of any plausible or counterfactual premise.

The paper is structured as follows. Section 2 outlines the formal preliminaries of propositional logic and the preferential semantic approach to conditionals on which our work is based. Section 3 is the heart of the paper. It describes the language of contextual conditionals, furnishes it with an appropriate and intuitive semantics, and motivates the corresponding logic by way of examples, formal properties, and a formal representation result. With the basics of the logic in place, Section 4 defines a form of entailment for it that is based on the well-known notion of *Rational Closure* (Lehmann and Magidor 1992), and shows that it is reducible to classical propositional reasoning. Section 5 reviews related work, while Section 6 concludes and considers future avenues to explore.

2 Formal background

In this paper, we assume a finite set of propositional *atoms* $\mathcal{P}$ and use $p, q, \ldots$ to denote its elements. Sentences of the underlying propositional language are denoted by $\alpha, \beta, \ldots$, and are built up from the atomic propositions and the

Boolean connectives in the usual way. The set of all propositional sentences is denoted by $\mathcal{L}$.

A *valuation* (alias *world*) is a function from $\mathcal{P}$ into $\{0, 1\}$. The set of all valuations is denoted $\mathcal{U}$, and we use $u, v, \ldots$ to denote its elements. Whenever it eases presentation, we represent valuations as sequences of atoms (e.g., p) and barred atoms (e.g., $\bar{\mathsf{p}}$), with the usual understanding. As an example, if $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$, with the atoms standing for, respectively, 'being a bird', 'being a flying creature', and 'being a penguin', the valuation $\mathsf{b}\bar{\mathsf{f}}\mathsf{p}$ conveys the idea that b is true, f is false, and p is true.

With $v \Vdash \alpha$ we denote the fact that v *satisfies* α. Given $\alpha \in \mathcal{L}$, with $[\![\alpha]\!] \stackrel{\text{def}}{=} \{v \in \mathcal{U} \mid v \Vdash \alpha\}$ we denote its *models*. For $X \subseteq \mathcal{L}$, $[\![X]\!] \stackrel{\text{def}}{=} \bigcap_{\alpha \in X} [\![\alpha]\!]$. We say $X \subseteq \mathcal{L}$ (classically) *entails* $\alpha \in \mathcal{L}$, denoted $X \models \alpha$, if $[\![X]\!] \subseteq [\![\alpha]\!]$. Given a set of valuations V, $\mathsf{fml}(V)$ indicates a formula characterising the set V. That is, $\mathsf{fml}(V)$ is a propositional formula satisfied by all and only the valuations in V.

A *defeasible conditional* $\mid\!\sim$ is a binary relation on $\mathcal{L}$ which is said to be *rational* (Kraus, Lehmann, and Magidor 1990) if it satisfies the well-known KLM properties below:

$$(\text{Ref})\quad \alpha \mid\!\sim \alpha \qquad\qquad (\text{LLE})\quad \frac{\models \alpha \leftrightarrow \beta,\ \alpha \mid\!\sim \gamma}{\beta \mid\!\sim \gamma}$$

$$(\text{And})\quad \frac{\alpha \mid\!\sim \beta,\ \alpha \mid\!\sim \gamma}{\alpha \mid\!\sim \beta \wedge \gamma} \qquad\qquad (\text{Or})\quad \frac{\alpha \mid\!\sim \gamma,\ \beta \mid\!\sim \gamma}{\alpha \vee \beta \mid\!\sim \gamma}$$

$$(\text{RW})\quad \frac{\alpha \mid\!\sim \beta,\ \models \beta \rightarrow \gamma}{\alpha \mid\!\sim \gamma} \qquad\qquad (\text{RM})\quad \frac{\alpha \mid\!\sim \beta,\ \alpha \not\mid\!\sim \neg\gamma}{\alpha \wedge \gamma \mid\!\sim \beta}$$

The merits of these properties have extensively been addressed in the literature (Gabbay 1984; Kraus, Lehmann, and Magidor 1990) and we shall not repeat them here.

A suitable semantics for rational conditionals is provided by ordered structures called *ranked interpretations*.

Definition 1. *A* ***ranked interpretation*** $\mathscr{R}$ *is a function from* $\mathcal{U}$ *to* $\mathbb{N} \cup \{\infty\}$*, satisfying the following* ***convexity property****: for every* $i \in \mathbb{N}$*, if* $\mathscr{R}(u) = i$*, then, for every* j $0 \leq j < i$*, there is a* $u' \in \mathcal{U}$ *for which* $\mathscr{R}(u') = j$*.*

For a given ranked interpretation $\mathscr{R}$ and valuation v, we denote with $\mathscr{R}(v)$ the *rank of* v. The number $\mathscr{R}(v)$ indicates the degree of *atypicality* of v. So the valuations judged most typical are those with rank 0, while those with an infinite rank are judged so atypical as to be implausible. We can therefore partition the set $\mathcal{U}$ w.r.t. $\mathscr{R}$ into the set of *plausible* valuations $\mathcal{U}^f_{\mathscr{R}} \stackrel{\text{def}}{=} \{u \in \mathcal{U} \mid \mathscr{R}(u) \in \mathbb{N}\}$, and *implausible* valuations $\mathcal{U}^{\infty}_{\mathscr{R}} \stackrel{\text{def}}{=} \mathcal{U} \setminus \mathcal{U}^f_{\mathscr{R}}$. With $[\![i]\!]_{\mathscr{R}}$, for $i \in \mathbb{N} \cup \{\infty\}$, we indicate all the valuations with rank i in $\mathscr{R}$ (we shall omit the subscript whenever it is clear from the context).

Assuming $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$, with the intuitions as above, Figure 1 below shows an example of a ranked interpretation.

Let $\mathscr{R}$ be a ranked interpretation and let $\alpha \in \mathcal{L}$. Then $[\![\alpha]\!]^f_{\mathscr{R}} \stackrel{\text{def}}{=} \mathcal{U}^f_{\mathscr{R}} \cap [\![\alpha]\!]$, and $\min[\![\alpha]\!]^f_{\mathscr{R}} \stackrel{\text{def}}{=} \{u \in [\![\alpha]\!]^f_{\mathscr{R}} \mid \mathscr{R}(u) \leq \mathscr{R}(v)$ for all $v \in [\![\alpha]\!]^f_{\mathscr{R}}\}$. A defeasible conditional $\alpha \mid\!\sim \beta$ can be given an intuitive semantics in terms of ranked interpretations as follows: $\alpha \mid\!\sim \beta$ is *satisfied in* $\mathscr{R}$ (denoted $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$) if $\min[\![\alpha]\!]^f_{\mathscr{R}} \subseteq [\![\beta]\!]$, with $\mathscr{R}$ referred to as a *ranked model* of $\alpha \mid\!\sim \beta$. In the example in Figure 1, we have $\mathscr{R} \Vdash \mathsf{b} \mid\!\sim \mathsf{f}$, $\mathscr{R} \Vdash \neg(\mathsf{p} \rightarrow \mathsf{b}) \mid\!\sim \bot$, $\mathscr{R} \Vdash \mathsf{p} \mid\!\sim \neg\mathsf{f}$, $\mathscr{R} \not\Vdash \mathsf{f} \mid\!\sim \mathsf{b}$, and $\mathscr{R} \Vdash \mathsf{p} \wedge \neg\mathsf{b} \mid\!\sim \mathsf{b}$. It is easily verified that $\mathscr{R} \Vdash \neg\alpha \mid\!\sim \bot$ iff $\mathcal{U}^f_{\mathscr{R}} \subseteq [\![\alpha]\!]$. Hence we frequently abbreviate $\neg\alpha \mid\!\sim \bot$ as α.

∞	$\bar{\mathsf{b}}\bar{\mathsf{f}}\mathsf{p}$, $\bar{\mathsf{b}}\mathsf{f}\mathsf{p}$
2	$\mathsf{b}\mathsf{f}\mathsf{p}$
1	$\mathsf{b}\bar{\mathsf{f}}\bar{\mathsf{p}}$, $\mathsf{b}\bar{\mathsf{f}}\mathsf{p}$
0	$\bar{\mathsf{b}}\bar{\mathsf{f}}\bar{\mathsf{p}}$, $\bar{\mathsf{b}}\mathsf{f}\bar{\mathsf{p}}$, $\mathsf{b}\mathsf{f}\bar{\mathsf{p}}$

Figure 1: A ranked interpretation for $\mathcal{P} = \{\mathsf{b}, \mathsf{f}, \mathsf{p}\}$.

The correspondence between rational conditionals and ranked interpretations is formalised by the following representation result.

Theorem 1 (Lehmann & Magidor, 1992; Gärdenfors & Makinson, 1994)**.** *A defeasible conditional* $\mid\!\sim$ *is rational iff there is an* $\mathscr{R}$ *such that* $\alpha \mid\!\sim \beta$ *iff* $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$*.*

With the semantics of the language of defeasible conditionals specified, the next important question is what entailment looks like for this logic. Consider a *conditional knowledge base* $\mathcal{K}$ as a finite set of statements $\alpha \mid\!\sim \beta$, with $\alpha, \beta \in \mathcal{L}$. As an example, let $\mathcal{K} = \{\mathsf{b} \mid\!\sim \mathsf{f}, \mathsf{p} \rightarrow \mathsf{b}, \mathsf{p} \mid\!\sim \neg\mathsf{f}\}$. A *ranked model* of $\mathcal{K}$ is a ranked interpretation satisfying all statements in $\mathcal{K}$. The ranked interpretation in Figure 1 is a ranked model of the above $\mathcal{K}$.

While several notions of entailment for conditional knowledge bases have been explored in the literature on nonmonotonic reasoning (Booth et al. 2019; Casini et al. 2014; Casini, Meyer, and Varzinczak 2019; Giordano et al. 2012; Lehmann 1995; Weydert 2003), *rational closure* (Lehmann and Magidor 1992) is commonly seen as the baseline for an appropriate form of entailment in this context. Basically, it formalises the principle of *presumption of typicality* (Lehmann 1995, p. 63), which informally specifies that a situation (valuation) should be assumed to be as typical as possible w.r.t. background information in a given $\mathcal{K}$.

Several equivalent definitions of rational closure can be found in the literature. We give the following one, due to Giordano et al. (2015). Given a knowledge base $\mathcal{K}$, we first define a weak ordering over the set of ranked models of $\mathcal{K}$ by setting $\mathscr{R}_1 \preceq_{\mathcal{K}} \mathscr{R}_2$, if, for every $v \in \mathcal{U}$, $\mathscr{R}_1(v) \leq \mathscr{R}_2(v)$. The intuition behind the ordering $\preceq_{\mathcal{K}}$ is that $\mathscr{R}_1$ is lower than $\mathscr{R}_2$ if it is more typical against the background of $\mathcal{K}$. The rational closure of $\mathcal{K}$ is then defined via the unique minimal ranked model of $\mathcal{K}$.

Definition 2. *Let* $\mathcal{K}$ *be a conditional knowledge base, and let* $\mathscr{R}^{\mathcal{K}}_{RC}$ *be the minimum element of* $\preceq_{\mathcal{K}}$ *on ranked models of* $\mathcal{K}$*. The* ***rational closure (RC)*** *of* $\mathcal{K}$ *is the defeasible consequence relation* $\mid\!\sim^{\mathcal{K}}_{RC} \stackrel{\text{def}}{=} \{\alpha \mid\!\sim \beta \mid \mathscr{R}^{\mathcal{K}}_{RC} \Vdash \alpha \mid\!\sim \beta\}$*.*

Figure 1 shows the minimum ranked model of $\mathcal{K} = \{\mathsf{b} \mid\!\sim \mathsf{f}, \mathsf{p} \rightarrow \mathsf{b}, \mathsf{p} \mid\!\sim \neg\mathsf{f}\}$ w.r.t. $\preceq_{\mathcal{K}}$. Hence we have that $\neg\mathsf{f} \mid\!\sim \neg\mathsf{b}$ is in the RC of $\mathcal{K}$.

3 Contextual conditionals

We now turn to the heart of the paper, the presentation of a logic for contextual conditionals. For a more detailed motivation, let us return to a more technical version of the penguin-dodo example in Section 1.

Example 2. *We know that birds usually fly (*$\mathsf{b} \mid\!\sim \mathsf{f}$*), that penguins are birds (*$\mathsf{p} \rightarrow \mathsf{b}$*) that usually do not fly (*$\mathsf{p} \mid\!\sim \neg\mathsf{f}$*). Also, we know that dodos were birds (*$\mathsf{d} \rightarrow \mathsf{b}$*) that usually did not fly (*$\mathsf{d} \mid\!\sim \neg\mathsf{f}$*), and that dodos do not exist anymore. Using the standard ranked semantics (Definition 1) we have two ways of modelling this information.*

The first option is to formalise what an agent believes by referring to valuations with rank 0 *in a ranked interpretation. That is, the agent believes* α *is true iff* $\top \mid\!\sim \alpha$ *holds. In such a case,* $\top \mid\!\sim \neg\mathsf{d}$ *means that the agent believes that dodos do not exist. The minimal model for this conditional knowledge base is shown in Figure 2 (left). The main limitation of this representation is that all exceptional entities have the same status as dodos, since they cannot be satisfied at rank* 0*. Hence we have* $\top \mid\!\sim \neg\mathsf{p}$*, just as we have* $\top \mid\!\sim \neg\mathsf{d}$*, and we are not able to distinguish between the status of the dodos (they do not exist anymore) and the status of the penguins (they are simply exceptional birds).*

The second option is to represent what an agent believes in terms of all valuations with finite ranks. That is, an agent believes α *to hold iff* $\neg\alpha \mid\!\sim \bot$ *holds. If dodos do not exist, we add the statement* $\mathsf{d} \mid\!\sim \bot$*. The minimal model for this case is depicted in Figure 2 (right). Here we can distinguish between what is considered false (dodos exist) and what is exceptional (penguins), but we are unable to reason coherently about counterfactuals, since from* $\mathsf{d} \mid\!\sim \bot$ *we can conclude anything about dodos.*

∞	$\mathcal{U} \setminus (\llbracket 0 \rrbracket \cup \llbracket 1 \rrbracket \cup \llbracket 2 \rrbracket)$
2	$\mathsf{p\overline{d}bf}$, $\mathsf{\overline{p}dbf}$, pdbf
1	$\mathsf{\overline{p}\overline{d}b\overline{f}}$, $\mathsf{p\overline{d}b\overline{f}}$, $\mathsf{\overline{p}db\overline{f}}$, $\mathsf{pdb\overline{f}}$
0	$\mathsf{\overline{p}\overline{d}bf}$, $\mathsf{\overline{p}\overline{d}\overline{b}f}$, $\mathsf{\overline{p}\overline{d}\overline{b}\overline{f}}$

∞	$\mathcal{U} \setminus (\llbracket 0 \rrbracket \cup \llbracket 1 \rrbracket \cup \llbracket 2 \rrbracket)$
2	$\mathsf{p\overline{d}bf}$
1	$\mathsf{\overline{p}\overline{d}b\overline{f}}$, $\mathsf{p\overline{d}b\overline{f}}$,
0	$\mathsf{\overline{p}\overline{d}bf}$, $\mathsf{\overline{p}\overline{d}\overline{b}f}$, $\mathsf{\overline{p}\overline{d}\overline{b}\overline{f}}$

Figure 2: Left: minimal ranked interpretation of the KB in Example 2 satisfying $\top \mid\!\sim \neg\mathsf{d}$. Right: minimal ranked interpretation of the KB expanded with $\mathsf{d} \mid\!\sim \bot$.

A *contextual conditional* (CC for short) is a statement of the form $\alpha \mid\!\sim_\gamma \beta$, with $\alpha, \beta, \gamma \in \mathcal{L}$, which is read as 'in the context of γ, β holds on condition that α holds'. Formally, a contextual conditional $\mid\!\sim$ is a ternary relation on $\mathcal{L}$. We shall write $\alpha \mid\!\sim_\gamma \beta$ as an abbreviation for $\langle \alpha, \beta, \gamma \rangle \in \mid\!\sim$.

To provide a suitable semantics for CCs we define a refined version of the ranked interpretations of Section 2 that we refer to as *epistemic interpretations*. A ranked interpretation can differentiate between plausible valuations (those in $\mathcal{U}^f_{\mathscr{R}}$) but not between implausible ones (those in $\mathcal{U}^\infty_{\mathscr{R}}$). In contrast, an epistemic interpretation can also differentiate between implausible valuations. We thus distinguish between two classes of valuations: plausible valuations with a *finite rank*, and implausible valuations with an *infinite rank*. Within implausible valuations we further distinguish between those that would be considered as *possible*, and those that would be *impossible*. This is formalised by assigning to each valuation u a tuple of the form $\langle f, i \rangle$ where $i \in \mathbb{N}$, or $\langle \infty, i \rangle$ where $i \in \mathbb{N} \cup \{\infty\}$. The f in $\langle f, i \rangle$ is intended to indicate that u has a *finite rank*, while the ∞ in $\langle \infty, i \rangle$ is intended to indicated that u has an *infinite rank*, where finite ranks are viewed as more typical than infinite ranks. Implausible valuations that are considered possible have an infinite rank $\langle \infty, i \rangle$ where $i \in \mathbb{N}$, while those considered impossible have the infinite rank $\langle \infty, \infty \rangle$, where $\langle \infty, \infty \rangle$ is taken to be less typical than any of the other infinite ranks.

To capture this formally, let $R \stackrel{\text{def}}{=} \{\langle f, i \rangle \mid i \in \mathbb{N}\} \cup \{\langle \infty, i \rangle \mid i \in \mathbb{N} \cup \{\infty\}\}$. We define the total ordering $\preceq$ over R as follows: $\langle x_1, y_1 \rangle \preceq \langle x_2, y_2 \rangle$ if and only if $x_1 = x_2$ and $y_1 \leq y_2$, or $x_1 = f$ and $x_2 = \infty$, where $i < \infty$ for all $i \in \mathbb{N}\}$. Before we can define epistemic interpretations, we need to extend the notion of convexity of ranked interpretations (Definition 1) to epistemic interpretations. Let e be a function from $\mathcal{U}$ to R. e is said to be *convex* (w.r.t.$\preceq$) if and only the following holds: i) If $e(u) = \langle f, i \rangle$, then, for all j s.t. $0 \leq j < i$, there is a $u_j \in \mathcal{U}$ s.t. $e(u_j) = \langle f, j \rangle$; and ii) if $e(u) = \langle \infty, i \rangle$ for $i \in \mathbb{N}$, then, for all j s.t. $0 \leq j < i$, there is a $u_j \in \mathcal{U}$ s.t. $e(u_j) = \langle \infty, j \rangle$.

Definition 3. *An **epistemic interpretation** E is a total function from* $\mathcal{U}$ *to R that is convex.*

Casini et al. (2020) have a similar definition of epistemic interpretations, but they do not allow for the rank $\langle \infty, \infty \rangle$.

We let $\mathcal{U}^f_E \stackrel{\text{def}}{=} \{u \in \mathcal{U} \mid E(u) = \langle f, i \rangle \text{ for some } i \in \mathbb{N}\}$ and $\mathcal{U}^\infty_E \stackrel{\text{def}}{=} \{u \in \mathcal{U} \mid E(u) = \langle \infty, i \rangle \text{ for some } i \in \mathbb{N}\}$. Note that $\mathcal{U}^\infty_E$ does *not* contain valuations with a rank of $\langle \infty, \infty \rangle$. We let $\min\llbracket \alpha \rrbracket_E \stackrel{\text{def}}{=} \{u \in \llbracket \alpha \rrbracket \mid E(u) \preceq E(v) \text{ for all } v \in \llbracket \alpha \rrbracket\}$, $\min\llbracket \alpha \rrbracket^f_E \stackrel{\text{def}}{=} \{u \in \llbracket \alpha \rrbracket \cap \mathcal{U}^f_E \mid E(u) \preceq E(v) \text{ for all } v \in \llbracket \alpha \rrbracket \cap \mathcal{U}^f_E\}$, and $\min\llbracket \alpha \rrbracket^\infty_E \stackrel{\text{def}}{=} \{u \in \llbracket \alpha \rrbracket \cap \mathcal{U}^\infty_E \mid E(u) \preceq E(v) \text{ for all } v \in \llbracket \alpha \rrbracket \cap \mathcal{U}^\infty_E\}$.

Observe that epistemic interpretations are allowed to have no plausible valuations ($\mathcal{U}^f_E = \emptyset$), as well as no implausible valuations that are possible ($\mathcal{U}^\infty_E = \emptyset$). This means it is possible that $E(u) = \langle \infty, \infty \rangle$ for all $u \in \mathcal{U}$, in which case $E \Vdash \alpha \mid\!\sim_\gamma \beta$ for all α, β, γ.

This allows us to provide a semantic definition of contextual conditionals in terms of epistemic interpretations.

Definition 4. $E \Vdash \alpha \mid\!\sim_\gamma \beta$ *(abbreviated as* $\alpha \mid\!\sim^E_\gamma \beta$*) if*

$$\begin{cases} \min\llbracket \alpha \wedge \gamma \rrbracket^f_E \subseteq \llbracket \beta \rrbracket & \textit{if } \llbracket \gamma \rrbracket \cap \mathcal{U}^f_E \neq \emptyset; \\ \min\llbracket \alpha \wedge \gamma \rrbracket^\infty_E \subseteq \llbracket \beta \rrbracket & \textit{otherwise.} \end{cases}$$

Intuitively, this definition evaluates $\alpha \mid\!\sim_\gamma \beta$ as follows. If the context γ is compatible with the plausible part of E (the valuations in $\mathcal{U}^f_E$) then $\alpha \mid\!\sim_\gamma \beta$ holds if the most typical plausible models of $\alpha \wedge \gamma$ are also models of β. On the other hand if the context γ is not compatible with the plausible part of E (that is, all models of γ have an infinite rank) then $\alpha \mid\!\sim_\gamma \beta$ holds if the most typical implausible (but possible) models of $\alpha \wedge \gamma$ are also models of β.

An immediate corollary of this is that the rational conditionals defined in terms of ranked interpretations can be simulated with CCs by setting the context to $\top$.

Definition 5. *For an epistemic interpretation E we define the ranked interpretation* $\mathscr{R}^E$ *extracted from E as follows:*

for $u \in \mathcal{U}_E^f$, $\mathscr{R}(u) = i$ where $E(u) = \langle f, i \rangle$ and $\mathscr{R}(u) = \infty$ for $u \in \mathcal{U} \setminus \mathcal{U}_E^f$.

Corollary 1. *Let E be an epistemic interpretation. Then $\mathscr{R}^E \Vdash \alpha \mathrel{|\!\!\sim} \beta$ iff $E \Vdash \alpha \mathrel{|\!\!\sim}_\top \beta$.*

The principal advantage of contextual conditionals and their associated enriched semantics in terms of epistemic interpretations is that it allows us to represent different degrees of epistemic involvement, with the finite ranks (the plausible valuations) representing the expectations of an agent. So $\top \mathrel{|\!\!\sim}_\top \alpha$ being true in E indicates that α is expected. What an agent believes to be true is what is true in all the valuations with finite ranks. That is, the agent believes α to be true iff $E \Vdash \neg\alpha \mathrel{|\!\!\sim}_\top \bot$. It is also possible to reason counterfactually. We can express that dodos would not fly, if they existed, in a coherent way. We can talk about dodos in a counterfactual context, for example assuming that Mauritius had never been colonised (mc): the conditional $\mathsf{d} \mathrel{|\!\!\sim}_{\neg\mathsf{mc}} \neg\mathsf{f}$ is read as 'In the context of Mauritius not having been colonised, the dodo would not fly'. Moreover, we can reason coherently with a contextual conditional, not even knowing whether its premises are plausible or counterfactual. To do so, it is sufficient to introduce statements of the form $\alpha \mathrel{|\!\!\sim}_\alpha \beta$. If α is plausible, this conditional is evaluated in the context of the finite ranks, exactly as if $\alpha \mathrel{|\!\!\sim}_\top \beta$ were being evaluated. On the other hand, if $\alpha \mathrel{|\!\!\sim}_\top \bot$ holds, $\alpha \mathrel{|\!\!\sim}_\alpha \beta$ will be evaluated referring to the infinite ranks. So, in the case of penguins and dodos, $\mathsf{p} \mathrel{|\!\!\sim}_\mathsf{p} \neg\mathsf{f}$ and $\mathsf{d} \mathrel{|\!\!\sim}_\mathsf{d} \neg\mathsf{f}$, express the information that penguins usually fly in the context of penguins existing, and that dodos usually fly in the context of dodos existing, regardless of whether the agent is aware of penguins or dodos existing or not. In contrast, a statement such as $\mathsf{d} \mathrel{|\!\!\sim}_\top \neg\mathsf{f}$ cannot be used to reason counterfactually about dodos. Note that it is still possible to impose that something necessarily holds. The conditional $\alpha \mathrel{|\!\!\sim}_\alpha \bot$ holds only in epistemic interpretations in which all models of α have $\langle\infty, \infty\rangle$ as their rank. The following example demomonstrates this more concretely.

Example 3. *Consider the following rephrasing of the statements in Example 2. 'Birds usually fly' becomes $\mathsf{b} \mathrel{|\!\!\sim}_\top \mathsf{f}$. Defeasible information about penguins and dodos are modelled using $\mathsf{p} \mathrel{|\!\!\sim}_\mathsf{p} \neg\mathsf{f}$ and $\mathsf{d} \mathrel{|\!\!\sim}_\mathsf{d} \neg\mathsf{f}$. Given that dodos don't exist anymore, the statement $\mathsf{d} \mathrel{|\!\!\sim}_\top \bot$ leaves open the existence of dodos in the infinite rank, which allows for coherent reasoning under the assumption that dodos exist (the context d). Moreover, information such as dodos and penguins necessarily being birds can be modelled by the conditionals $\mathsf{p} \wedge \neg\mathsf{b} \mathrel{|\!\!\sim}_{\mathsf{p}\wedge\neg\mathsf{b}} \bot$ and $\mathsf{d} \wedge \neg\mathsf{b} \mathrel{|\!\!\sim}_{\mathsf{d}\wedge\neg\mathsf{b}} \bot$, relegating the valuations in $[\![\mathsf{p} \wedge \neg\mathsf{b}]\!] \cup [\![\mathsf{d} \wedge \neg\mathsf{b}]\!]$ to the rank $\langle\infty, \infty\rangle$. Figure 3 shows a model of these statements.*

Next we consider the class of contextual conditionals from the perspective of a list of *contextual* rationality properties in the KLM style. We start with the following ones:

$\langle\infty, \infty\rangle$	$[\![\mathsf{p} \wedge \neg\mathsf{b}]\!] \cup [\![\mathsf{d} \wedge \neg\mathsf{b}]\!]$
$\langle\infty, 1\rangle$	$\overline{\mathsf{p}}\mathsf{dbf}$, pdbf
$\langle\infty, 0\rangle$	$\overline{\mathsf{p}}\mathsf{db}\overline{\mathsf{f}}$, $\mathsf{pdb}\overline{\mathsf{f}}$
$\langle f, 2\rangle$	$\mathsf{p}\overline{\mathsf{d}}\mathsf{bf}$
$\langle f, 1\rangle$	$\overline{\mathsf{p}}\overline{\mathsf{d}}\mathsf{b}\overline{\mathsf{f}}$, $\mathsf{p}\overline{\mathsf{d}}\mathsf{b}\overline{\mathsf{f}}$
$\langle f, 0\rangle$	$\overline{\mathsf{p}}\overline{\mathsf{d}}\mathsf{bf}$, $\overline{\mathsf{p}}\overline{\mathsf{d}}\overline{\mathsf{b}}\mathsf{f}$, $\overline{\mathsf{p}}\overline{\mathsf{d}}\overline{\mathsf{b}}\overline{\mathsf{f}}$

Figure 3: Model of the statements in Example 3.

$$\text{(Ref)}\quad \alpha \mathrel{|\!\!\sim}_\gamma \alpha \qquad\qquad \text{(LLE)}\quad \frac{\models \alpha \leftrightarrow \beta,\ \alpha \mathrel{|\!\!\sim}_\gamma \delta}{\beta \mathrel{|\!\!\sim}_\gamma \delta}$$

$$\text{(And)}\quad \frac{\alpha \mathrel{|\!\!\sim}_\gamma \beta,\ \alpha \mathrel{|\!\!\sim}_\gamma \delta}{\alpha \mathrel{|\!\!\sim}_\gamma \beta \wedge \delta} \qquad\qquad \text{(Or)}\quad \frac{\alpha \mathrel{|\!\!\sim}_\gamma \delta,\ \beta \mathrel{|\!\!\sim}_\gamma \delta}{\alpha \vee \beta \mathrel{|\!\!\sim}_\gamma \delta}$$

$$\text{(RW)}\quad \frac{\alpha \mathrel{|\!\!\sim}_\gamma \beta,\ \models \beta \rightarrow \delta}{\alpha \mathrel{|\!\!\sim}_\gamma \delta} \qquad\qquad \text{(RM)}\quad \frac{\alpha \mathrel{|\!\!\sim}_\gamma \beta,\ \alpha \not\mathrel{|\!\!\sim}_\gamma \neg\delta}{\alpha \wedge \delta \mathrel{|\!\!\sim}_\gamma \beta}$$

Observe that they correspond exactly to the original KLM properties, except that context has been added.

Definition 6. *A CC $\mathrel{|\!\!\sim}$ is* basic *(a BCC) if it satisfies the contextual KLM properties.*

An immediate corollary of this definition is that for a BCC with the context γ fixed, $\mathrel{|\!\!\sim}_\gamma$ is a rational conditional. We then get the following result.

Theorem 2. *Every epistemic interpretation generates a BCC, but the converse does not hold.*

The reason why the converse of Theorem 2 does not hold is that the structure of a BCC is completely independent of the context γ referred to in the contextual KLM properties. As a very simple instance of this problem, observe that BCCs are not even syntax-dependent w.r.t. the context. That is, we may have $\alpha \mathrel{|\!\!\sim}_\gamma \beta$ but $\alpha \not\mathrel{|\!\!\sim}_\delta \beta$ where $\gamma \equiv \delta$. To remedy this, we require BCCs to satisfy the following additional properties:

$$\text{(Inc)}\quad \frac{\alpha \mathrel{|\!\!\sim}_\gamma \beta}{\alpha \wedge \gamma \mathrel{|\!\!\sim}_\top \beta} \qquad\qquad \text{(Vac)}\quad \frac{\top \not\mathrel{|\!\!\sim}_\top \neg\gamma,\ \alpha \wedge \gamma \mathrel{|\!\!\sim}_\top \beta}{\alpha \mathrel{|\!\!\sim}_\gamma \beta}$$

$$\text{(Ext)}\quad \frac{\gamma \equiv \delta}{\alpha \mathrel{|\!\!\sim}_\gamma \beta \text{ iff } \alpha \mathrel{|\!\!\sim}_\delta \beta} \qquad\qquad \text{(SupExp)}\quad \frac{\alpha \mathrel{|\!\!\sim}_{\gamma\wedge\delta} \beta}{\alpha \wedge \gamma \mathrel{|\!\!\sim}_\delta \beta}$$

$$\text{(SubExp)}\quad \frac{\delta \mathrel{|\!\!\sim}_\top \bot,\ \alpha \wedge \gamma \mathrel{|\!\!\sim}_\delta \beta}{\alpha \mathrel{|\!\!\sim}_{\gamma\wedge\delta} \beta}$$

We refer to these as the *contextual AGM properties* for reasons to be outlined below.

Definition 7. *A BCC is a full CC (FCC) if it satisfies the contextual AGM properties.*

One way in which to interpret the addition of a context to conditionals from a technical perspective, is to think of it as similar to *belief revision*. That is, $\alpha \mathrel{|\!\!\sim}_\gamma \beta$ can be thought of stating that if a revision with γ has taken place, then β will hold on condition that α holds. With this view of contextual conditionals, the contextual AGM properties above are seen as versions of the AGM properties for belief revision (Alchourrón, Gärdenfors, and Makinson 1985). The names of these properties were chosen with the names of their AGM analogues in mind. The contextual AGM properties can be motivated intuitively as follows.

Together, Inc and Vac require that when the context (or revision with) γ is compatible with what is currently plausible, then a conditional w.r.t. the context γ (a 'revison by' γ) is the same as a conditional where the context is $\top$ (where there isn't a 'revision' at all), but with γ added to the premise of the conditional. Ext ensures that context is syntax-independent. Finally, (SupExp) and (SubExp) together require that if the context δ is implausible (that is, the 'revision' with δ is incompatible with what is plausible) then a conditional w.r.t. the context $\gamma \wedge \delta$ (a 'revision by' $\gamma \wedge \delta$) is the same as a conditional where the context (or 'revision') is δ, but with γ added to the premise of the conditional.

It turns out that FCCs are characterised by epistemic interpretations, resulting in the following representation result.

Theorem 3. *Every epistemic interpretation generates an FCC. Every FCC can be generated by an epistemic interpretation.*

The AGM-savvy reader may have noticed that the following two obvious analogues of the suite of contextual AGM properties are missing from our list above.

(Succ) $\alpha \mid\!\sim_\gamma \gamma$

(Cons) $\top \mid\!\sim_\gamma \bot$ iff $\gamma \equiv \bot$

Succ requires context to matter: a 'revision' by γ will always be successful. Cons states that we will obtain an inconsistency only when the context is inconsistent.

It turns out that Succ holds for epistemic interpretations, but follows from the combination of the contextual KLM and AGM properties, while Cons does not.

Corollary 2. *Every FCC satisfies Succ, but there are FCCs for which Cons does not hold.*

The fact that Cons does not hold can be explained by considering the epistemic interpretation where all valuations are taken to be impossible (that is, to have the rank $\langle\infty, \infty\rangle$) in which case all statements of the form $\alpha \mid\!\sim_\gamma \beta$ are true.

We conclude this section by considering the following two properties.

(Incons) $\alpha \mid\!\sim_\bot \beta$

(Cond) If $\gamma \not\mid\!\sim_\top \bot$ then $\alpha \wedge \gamma \mid\!\sim_\top \beta$ iff $\alpha \mid\!\sim_\gamma \beta$

Incons requires that all conditionals hold when the context is inconsistent, while Cond requires that conditionals w.r.t. the context γ be equivalent to the same conditional with γ added to the premise whenever there is no context (when the context is $\top$).

Proposition 1. *Every FCC satisfies Incons and Cond.*

4 Entailment

Up to this point we have investigated the properties characterising the class of epistemic interpretations. Here we move to investigating how we can reason in this framework. That is, given a knowledge base (a finite set) of contextual conditionals (CCKB), what new contextual conditionals are we justified in inferring? As widely discussed in the KLM and other analogous conditional approaches, in the non-monotonic framework it is generally not useful to define an entailment relation with a Tarskian approach, that is, taking under consideration what holds in *all* the models of a KB, since the resulting entailment relation is too weak inferentially (Lehmann and Magidor 1992). More interesting entailment relations can be defined by picking a single model of the KB. It is generally accepted that there are many appropriate entailment relations that can be defined for defeasible reasoning, depending on the kind of reasoning we want to model (Lehmann 1995; Casini, Meyer, and Varzinczak 2019), but in the framework of preferential semantics the RC, recalled in Section 2, is generally recognised as a basic construction, from the refinement of which we can obtain other interesting entailment relation.

We now present a reformulation of the same kind of construction in our framework, that we call *Minimal Closure* (MC). We adapt to our framework the notion of a minimal model (Giordano et al. 2015), recalled in Section 2, and we show that for any CCKB the minimal model is unique.

The construction of the minimal model will be obtained creating a bridge between contextual conditionals and epistemic interpretations on one hand and defeasible conditionals and ranked interpretations on the other. Some notions can be naturally extended from the latter framework to the former one. First of all, we can extend the notion of consistency. A set $\mathcal{C}$ of defeasible conditionals is *consistent* iff it has a ranked model $\mathscr{R}$ s.t. $[\![0]\!]_{\mathscr{R}} \neq \emptyset$. This is the case since such a model does not satisfy the conditional $\top \mid\!\sim \bot$, that represents absurdity in the conditional framework. This condition can be easily translated into our framework.

Definition 8. *A CCKB $\mathcal{K}$ is consistent iff it has an epistemic model E s.t. $[\![\langle f, 0\rangle]\!]_E \neq \emptyset$.*

That is, A CCKB $\mathcal{K}$ is consistent iff it has an epistemic model *E* that does not satisfy $\top \mid\!\sim_\top \bot$. $[\![\langle f, 0\rangle]\!]_E$ is a notation for epistemic interpretations that mirrors the notation $[\![0]\!]_R$ for ranked interpretations, that is, $[\![\langle x, y\rangle]\!]_E$ represents the set of worlds that have rank $\langle x, y\rangle$ in *E*.

Given Corollary 1, we can define the satisfaction of defeasible conditionals also for epistemic interpretations:

$$E \Vdash \alpha \mid\!\sim \beta \text{ iff } E \Vdash \alpha \mid\!\sim_\top \beta$$

Note that an epistemic interpretation *E* satisfies exactly the same defeasible conditionals of its extracted ranked interpretation $\mathscr{R}^E$ (see Definition 5). That is, the ranks specified inside $\mathcal{U}_E^\infty \cup [\![\langle\infty, \infty\rangle]\!]$ are totally irrelevant w.r.t. the satisfaction of the defeasible conditionals $\alpha \mid\!\sim \beta$. We can also intuitively define the converse operation of the extraction of a ranked interpretation from an epistemic interpretation: we can *generate* an epistemic interpretation from a ranked interpretation.

Definition 9. *For a ranked interpretation $\mathscr{R}$ we define the epistemic interpretation $E^{\mathscr{R}}$ extracted from E as follows: for $u \in \mathcal{U}^f_{\mathscr{R}}$, $E^{\mathscr{R}}(u) = \langle f, i\rangle$ where $\mathscr{R}(u) = i$ and $E^{\mathscr{R}}(u) = \langle\infty, \infty\rangle$ for $u \in \mathcal{U} \setminus \mathcal{U}^f_{\mathscr{R}}$.*

It is easy to see that $\mathscr{R}$ and $E^{\mathscr{R}}$ are equivalent w.r.t. the satisfaction of defeasible conditionals.

The following corollary of Proposition 1,that is simply a semantic reformulation of the property (Cond), will be central in connecting the satisfaction of contextual conditionals to the satisfaction of the defeasible ones.

Corollary 3. *For any epistemic interpretation E, if $\mathcal{U}_E^f \cap [\![\gamma]\!] \neq \emptyset$ then $E \Vdash \alpha \mid\!\sim_\gamma \beta$ iff $E \Vdash \alpha \wedge \gamma \mid\!\sim \beta$.*

Given Corollary 3, we define a simple transformation: given a CCKB $\mathcal{K}$, let $\mathcal{K}^\wedge$ be its *conjunctive classical form*:

$$\mathcal{K}^\wedge = \{\alpha \wedge \gamma \mid\!\sim \beta \mid \alpha \mid\!\sim_\gamma \beta \in \mathcal{K}\}.$$

We can use the conjunctive classical form to define two relevant models for a CCKB $\mathcal{K}$: the *classical epistemic model* and the *minimal epistemic model*. The former is the epistemic interpretation generated by the minimal ranked model of $\mathcal{K}^\wedge$.

Definition 10 (classical epistemic model)**.** *Let $\mathcal{K}$ be a CCKB, $\mathcal{K}^\wedge$ its conjunctive classical form, and $\mathscr{R}$ the minimal ranked model of $\mathcal{K}^\wedge$. The classical epistemic model of $\mathcal{K}$ is the epistemic interpretation $E^{\mathscr{R}}$ generated from $\mathscr{R}$.*

Since $\mathscr{R}$ is a ranked model of $\mathcal{K}^\wedge$, also $E^{\mathscr{R}}$ is. We need to check that it is also a ranked model of $\mathcal{K}$.

Proposition 2. *Let $\mathcal{K}$ be a conditional base, and let $E^{\mathscr{R}}$ be defined as in Definition 10. $E^{\mathscr{R}}$ is a model of $\mathcal{K}$.*

The proof is immediate, given Corollary 3. From Proposition 2 and Corollary 3 we can also easily prove the following.

Proposition 3. *Let $\mathcal{K}$ be a conditional base. $\mathcal{K}$ has an epistemic model iff $\mathcal{K}^\wedge$ has a ranked model.*

By linking the satisfaction of a CCKB $\mathcal{K}$ to the satisfaction of its conjunctive form $\mathcal{K}^\wedge$ we are able to define a simple method to check the consistency of a CCKB, based on the *materialisation* $\overline{\mathcal{K}^\wedge}$ of $\mathcal{K}^\wedge$. The materialisation $\overline{\mathcal{C}}$ of a set of defeasible conditionals $\mathcal{C}$ is the set of material implications corresponding to the conditionals in $\mathcal{C}$: $\overline{\mathcal{C}} \stackrel{\text{def}}{=} \{\alpha \rightarrow \beta \mid \alpha \mid\!\sim \beta \in \mathcal{C}\}$.

Corollary 4. *A CCKB $\mathcal{K}$ is consistent iff $\overline{\mathcal{K}^\wedge} \not\models \bot$.*

This corollary is immediate from Proposition 3 and the well-known property that a finite set of defeasible conditionals is consistent iff its materialisation is a consistent propositional knowledge base (Lehmann and Magidor 1992, Lemma 5.21).

A classical epistemic model is a direct translation of a ranked interpretation into an equivalent epistemic interpretation, and it is useful to prove how a consistency check can be reduced to a simple propositional check. However, since it does not go beyond the modelling possibilities of ranked interpretations, this model is not appropriate to define an interesting form of entailment. Hence we now move to the definition of the *minimal epistemic model*, referring to the minimality order introduced for ranked interpretations in Section 2.

We need to adapt, in an intuitive way, the notion of minimality defined for the ranked interpretations (Giordano et al. 2015) to the present framework. In Section 3 we defined a total ordering $\preceq$ over the tuples $\langle x, y\rangle$ representing the ranks in epistemic interpretations. Let the ordering $\prec_{\mathcal{K}}$ on all the epistemic models of a CCKB $\mathcal{K}$ be defined as follows: $E_1 \prec_{\mathcal{K}} E_2$, if, for every $v \in \mathcal{U}$, $E_1(v) \preceq E_2(v)$, and there is a $w \in \mathcal{U}$ s.t. $E_2(w) \not\preceq E_1(w)$.

Definition 11. *Let $\mathcal{K}$ be a consistent CCKB, and $\mathcal{E}_{\mathcal{K}}$ be the set of its epistemic models. $E \in \mathcal{E}_{\mathcal{K}}$ is a minimal epistemic model of $\mathcal{K}$ iff there is no $E' \in \mathcal{E}_{\mathcal{K}}$ s.t. $E' \prec_{\mathcal{K}} E$.*

We first define a construction of a model, given a consistent CCKB $\mathcal{K}$. Then we prove that it is actually the unique minimal epistemic model of $\mathcal{K}$.

Definition 12 (minimal epistemic model)**.** *Let $\mathcal{K}$ be a consistent CCKB, $\mathcal{K}^\wedge$ its conjunctive classical form, and $\mathscr{R}$ be the minimal ranked model of $\mathcal{K}^\wedge$. We identify the conditionals in $\mathcal{K}$ with a context that has infinite rank in $\mathscr{R}$.*

- $\mathcal{K}_\infty \stackrel{\text{def}}{=} \{\alpha \mid\!\sim_\gamma \beta \in \mathcal{K} \mid \mathscr{R}(\gamma) = \infty\}$;
- $\mathcal{K}^\wedge_{\infty\downarrow} \stackrel{\text{def}}{=} \{\alpha \wedge \gamma \mid\!\sim \beta \mid \alpha \mid\!\sim_\gamma \beta \in \mathcal{K}_\infty\} \cup \{\mathsf{fml}(\mathcal{U}^f_{\mathscr{R}}) \mid\!\sim \bot\}$.

We construct the interpretation $E_{\mathcal{K}}$ in the following way:

1. *For every $u \in \mathcal{U}^f_{\mathscr{R}}$, if $\mathscr{R}(u) = i$, then $E_{\mathcal{K}}(u) = \langle f, i\rangle$;*
2. *Let $\mathscr{R}'$ be the the minimal ranked model of $\mathcal{K}^\wedge_{\infty\downarrow}$. For every $u \in \mathcal{U}^\infty_{\mathscr{R}}$, if $\mathscr{R}'(u) = i$, then $E_{\mathcal{K}}(u) = \langle \infty, i\rangle$.*

More informally, Definition 12 proceeds as follows. First we want to partition the contexts that can be satisfied in some plausible worlds from those with infinite rank. γ is not satisfiable in a plausible valuation iff $\gamma \mid\!\sim_\top \bot$ is satisfied in every model of $\mathcal{K}$, that, by Corollary 3 and Corollary 1, justifies the use of the minimal ranked model $\mathscr{R}$ of the conjunctive form $\mathcal{K}^\wedge$ for the identification of $\mathcal{K}_\infty$. We then identify the minimal configuration satisfying $\mathcal{K}$, considering first the finite ranks, and then the infinite ones. Corollary 3 tells us that, w.r.t. the plausible contexts, the minimal configuration is associated with the conjunctive normal form. Hence we refer again to the minimal ranked model $\mathscr{R}$ of $\mathcal{K}^\wedge$ to decide the configuration of the plausible valuations (Point 1). In order to configure the infinite ranks, the knowledge base $\mathcal{K}^\wedge_{\infty\downarrow}$ considers all the counterfactual conditionals in $\mathcal{K}_\infty$, and requires all plausible valuations in $\mathscr{R}$ to have an infinite rank. $\mathscr{R}'$ defines the minimal configuration that satisfies the conditionals in $\mathcal{K}^\wedge_{\infty\downarrow}$, and at Point 2 we put such a configuration "on top" of the finite ranks to define $E_{\mathcal{K}}$.

We need to prove that $E_{\mathcal{K}}$ is an epistemic model of $\mathcal{K}$, and that it is the unique minimal epistemic model of $\mathcal{K}$.

Let E be an epistemic interpretation. We can build an interpretation E^∞, the *counterfactual shifting* of E, as follows:

$$E^\infty_\downarrow(u) \stackrel{\text{def}}{=} \begin{cases} \langle f, i\rangle & \text{if } E(u) = \langle \infty, i\rangle \text{ with } i < \infty; \\ \langle \infty, \infty\rangle & \text{otherwise.} \end{cases}$$

Intuitively, $E^\infty_\downarrow$ simply shifts the infinite ranks in E to the finite ranks. For $E^\infty_\downarrow$ we can prove a lemma corresponding to Corollary 3.

Lemma 1. *For any epistemic interpretation E, if $\mathcal{U}^f_E \cap [\![\gamma]\!] = \emptyset$ then $E \Vdash \alpha \mid\!\sim_\gamma \beta$ iff $E^\infty_\downarrow \Vdash \alpha \wedge \gamma \mid\!\sim \beta$.*

Using Corollary 3 and Lemma 1, it is quite easy to prove that $E_{\mathcal{K}}$ is an epistemic model of $\mathcal{K}$.

Proposition 4. *Let $\mathcal{K}$ be a consistent CCKB, and let $E_{\mathcal{K}}$ be an epistemic interpretation built as in Definition 12. $E_{\mathcal{K}}$ is an epistemic model of $\mathcal{K}$.*

We proceed by showing that $E_{\mathcal{K}}$ is actually the only minimal epistemic model of $\mathcal{K}$.

Proposition 5. *Let $\mathcal{K}$ be a consistent CCKB, and let $E_{\mathcal{K}}$ be an epistemic interpretation built as in Definition 12. $E_{\mathcal{K}}$ is the only minimal epistemic model of $\mathcal{K}$.*

Also in this case, the proof rests on Definition 12, Corollary 3, and Lemma 1.

Example 4. *The model of the CCKB $\mathcal{K} = \{\mathsf{b} \mid\!\sim_{\top} \mathsf{f}, \mathsf{p} \mid\!\sim_{\mathsf{p}} \neg\mathsf{f}, \mathsf{d} \mid\!\sim_{\mathsf{d}} \neg\mathsf{f}, \mathsf{d} \mid\!\sim_{\top} \bot, (\mathsf{p} \wedge \neg\mathsf{b}) \mid\!\sim_{(\mathsf{p}\wedge\neg\mathsf{b})} \bot, (\mathsf{d} \wedge \neg\mathsf{b}) \mid\!\sim_{(\mathsf{d}\wedge\neg\mathsf{b})} \bot\}$ in Example 3, that is described in Figure 3, is the minimal epistemic model of the KB, obtained following Definition 12, where $\mathcal{K}_{\infty} = \{\mathsf{d} \mid\!\sim_{\mathsf{d}} \neg\mathsf{f}, (\mathsf{p}\wedge\neg\mathsf{b}) \mid\!\sim_{(\mathsf{p}\wedge\neg\mathsf{b})} \bot, (\mathsf{d} \wedge \neg\mathsf{b}) \mid\!\sim_{(\mathsf{d}\wedge\neg\mathsf{b})} \bot\}$.*

The minimal closure of $\mathcal{K}$ is defined in terms of this minimum epistemic model of $\mathcal{K}$.

Definition 13 (Minimal Closure)**.** *$\alpha \mid\!\sim_{\gamma} \beta$ is* minimally entailed *by a CCKB $\mathcal{K}$, indicated as $\mathcal{K} \models_m \alpha \mid\!\sim_{\gamma} \beta$, iff $E_{\mathcal{K}} \Vdash \alpha \mid\!\sim_{\gamma} \beta$, where $E_{\mathcal{K}}$ is the minimal model of $\mathcal{K}$. The correspondent closure operation*

$$\mathcal{C}_m(\mathcal{K}) \stackrel{\text{def}}{=} \{\alpha \mid\!\sim_{\gamma} \beta \mid \mathcal{K} \models_m \alpha \mid\!\sim_{\gamma} \beta\}$$

is the minimal closure *of $\mathcal{K}$.*

Example 5. *We proceed from Example 4. Looking at the model in Figure 3, we are able to check what is minimally entailed. For every $\alpha \mid\!\sim_{\gamma} \beta \in \mathcal{K}$, $\mathcal{K} \models_m \alpha \mid\!\sim_{\gamma} \beta$. In particular, while $\mathcal{K} \models_m d \mid\!\sim_{\top} \bot$, we do not have $\mathcal{K} \models_m d \mid\!\sim_{d} \bot$, that is, it is possible to reason counterfactually about dodos. From the point of view of the actual situation (that is, in the context $\top$) we can derive everything about dodos, since they do not exist: we have both $\mathcal{K} \models_m \mathsf{d} \mid\!\sim_{\top} \neg\mathsf{f}$ and $\mathcal{K} \models_m \mathsf{d} \mid\!\sim_{\top} \mathsf{f}$. However, we are able to reason coherently about dodos once we assume a point of view in which they would exist: we have in fact $\mathcal{K} \models_m \mathsf{d} \mid\!\sim_{\mathsf{d}} \neg\mathsf{f}$, but $\mathcal{K} \not\models_m \mathsf{d} \mid\!\sim_{\mathsf{d}} \mathsf{f}$.*

Definition 11 shows that the minimal epistemic model can be defined using the minimal ranked models for two sets of defeasible conditionals, $\mathcal{K}^{\wedge}$ and $\mathcal{K}^{\wedge}_{\infty\downarrow}$. That is, we do so using the RC of each one. Now, there are decision procedures for RC that fully rely on a series of propositional decision steps (Freund 1998; Casini and Straccia 2010). In short, which contextual conditionals hold in the minimal epistemic model can be decided by checking what holds in two minimal ranked models, and what holds in a minimal ranked model can be decided using a procedure that relies on propositional steps. Starting from this, it is also possible to define a decision procedure for $\models_m$ that fully relies on a series of propositional decision steps, that we omit here due to space limitations.

5 Related work

About the distinction between plausible and implausible state of affairs, a similar distinction has been used by Booth et al. (2014), where some pieces of information are considered *credible* while others are not.

The literature on the notion of context is vast, and several formalisations and applications of it have been studied across many areas within AI (Bikakis and Antoniou 2010; Ghidini and Giunchiglia 2001; Homola and Serafini 2012; Klarman and Gutiérrez-Basulto 2013; Pérez and Uzcátegui 1999).

The role of context in conditional-like statements has been explored recently, in particular in defeasible reasoning over description logic ontologies and within semantic frameworks that are closely related to ours. Britz and Varzinczak (2018; 2019), for example, have put forward a notion of defeasible class inclusion parameterised by atomic roles. Their semantics allows for multiple preference relations on objects, which is more general than our single-preference approach, and allows for objects to be compared in more than one way. This makes normality (or typicality) context dependent and gives more flexibility from a modelling perspective. Giordano and Gliozzi (2018) consider reasoning about multiple aspects in defeasible description logics where the notion of aspect (or context) is linked to concept names (alias, atoms) also in a multi-preference semantics.

When compared with our framework, neither of the above mentioned approaches allow for reasoning about objects that are 'forbidden' by the background knowledge. In that respect, our proposal is complementary to theirs and a contextual form of class inclusion along the lines of the ternary $\mid\!\sim$ here studied, with potential applications going beyond that of defeasible reasoning in ontologies, is worth exploring as future work.

6 Concluding remarks

The main contributions of the present paper can be summarised as follows: (*i*) the motivation for and the provision of a simple context-based form of conditional which is general enough to be used in several application domains, as our examples illustrate; (*ii*) an intuitive semantics which is based on a semantic construction that has proven useful in the area of belief change and that is more general and also more fine-grained than the standard preferential semantics; (*iii*) an investigation of the properties that contextual conditionals satisfy and of their appropriateness for knowledge representation and reasoning, in particular when reasoning about information that is incompatible with background knowledge, and (*iv*) the definition of a form of entailment for contextual conditional knowledge bases based on the widely-accepted notion of RC, which is reducible to classical propositional reasoning. Space considerations prevent us from presenting an algorithm for deciding entailment within our framework.

Next steps are the extension of this approach to other logics. Description Logics, for which RC has already been reformulated (Bonatti 2019; Casini and Straccia 2010; Giordano et al. 2015), are the first candidates. We also plan to investigate refinements of RC such as lexicographic closure (Lehmann 1995) and their variants (Casini et al. 2014; Casini, Meyer, and Varzinczak 2019; Casini and Straccia 2013).

Acknowledgments. This research was partially supported by TAILOR, a project funded by EU Horizon 2020 research and innovation programme under GA No 952215.

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