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Contextual rational closure for defeasible $\mathcal{ALC}$

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Abstract Description logics have been extended in a number of ways to support defeasible reasoning in the KLM tradition. Such features include preferential or rational defeasible concept inclusion, and defeasible roles in complex concept descriptions. Semantically, defeasible subsumption is obtained by means of a preference order on objects, while defeasible roles are obtained by adding a preference order to role interpretations. In this paper, we address an important limitation in defeasible extensions of description logics, namely the restriction in the semantics of defeasible concept inclusion to a single preference order on objects. We do this by inducing a modular preference order on objects from each modular preference order on roles, and using these to relativise defeasible subsumption. This yields a notion of contextualised rational defeasible subsumption, with contexts described by roles. We also provide a semantic construction for rational closure and a method for its computation, and present a correspondence result between the two.

Keywords Description logics, non-monotonic reasoning, defeasible subsumption, preferential semantics, rational closure, context

1 Introduction

Description Logics (DLs) [2] are decidable fragments of first-order logic that serve as the formal foundation for Semantic-Web ontologies. As witnessed by

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recent developments in the field, DLs still allow for meaningful, decidable extensions, as new knowledge representation requirements are identified. A case in point is the need to allow for exceptions and defeasibility in reasoning over logic-based ontologies [4–6, 15, 16, 19, 21, 23, 29, 31, 32, 37, 38, 44, 46]. Yet, DLs do not allow for the direct expression of and reasoning with different aspects of defeasibility.

Given the special status of concept inclusion in DLs in particular, and the historical importance of entailment in logic in general, past research efforts in this direction have focused primarily on accounts of defeasible subsumption and the characterisation of defeasible entailment. Semantically, the latter usually takes as point of departure orderings on a class of first-order interpretations [36, 38], whereas the former usually assume a preference order on objects of the domain [17, 18, 33].

Recently, we proposed decidable extensions of DLs supporting defeasible knowledge representation and reasoning over ontologies [21, 23, 24, 27]. Our proposal built on previous work to resolve two important ontological limitations of the preferential approach to defeasibility in DLs — the assumption of a single preference order on all objects in the domain of interpretation, and the assumption that defeasibility is intrinsically linked to arguments or conditionals [20, 22, 25, 26].

We achieved this by introducing non-monotonic reasoning features that any classical DL can be extended with in the concept language, in subsumption statements and in role assertions, via an intuitive notion of normality for roles [21]. This parameterised the idea of preference while at the same time introducing the notion of defeasible class membership. Defeasible subsumption allows for the expression of statements of the form "C is usually subsumed by D", for example, "Chenin blanc wines *are usually* unwooded". In the extended language, one can also refer directly to, for example, "Chenin blanc wines that *usually have* a wood aroma". We can also combine these seamlessly, as in: "Chenin blanc wines that *usually have* a wood aroma *are usually* wooded". This cannot be expressed in terms of defeasible subsumption alone, nor can it be expressed w.l.o.g. using typicality-based operators [7, 9, 33, 34, 48] on concepts. This is because the semantics of the expression is inextricably tied to the two distinct uses of the term 'usually', one applying to objects and the other to relationships.

Nevertheless, even this generalisation leaves open the question of different, possibly incompatible, notions of defeasibility in subsumption, similar to those studied in contextual argumentation [1, 3]. In the statement "Chenin blanc wines are usually unwooded", the context relative to which the subsumption is normal is left implicit — in this case, the style of the wine. In a different context such as consumer preference or origin, the most preferred (or normal, or typical) Chenin blanc wines may not correlate with the usual wine style. Wine x may be more exceptional than y in one context, but less exceptional in another context. This represents a form of inconsistency in defeasible knowledge bases that could arise from the presence of named individuals in the ontology. The example illustrates why a single ordering on individuals, as it is usually assumed, does not suffice. It also points to a natural index for relativised context, namely the use of preferential role names as we have previously proposed [21]. Using role names to indicate context in defeasible subsumption has the advantage that the vocabulary of the language does not have to be extended by a new set of context names. Furthermore, opting for role names rather than concept names to indicate context has a simpler semantics, since constructs to form complex roles are either absent or limited. The semantics of roles can also be suitably constrained by concept inclusions, for example by defining domain restrictions. In this paper, we therefore propose to induce multiple preference orders on objects from preference orders on role interpretations, and use these to relativise defeasible subsumption. This yields a notion of contextualised defeasible subsumption, with contexts indicated by role names.

The remainder of the present paper is structured as follows: In Section 2, we provide a summary of the required background on $\mathcal{ALC}$, the prototypical description logic and on which we shall focus in the present work. In Section 3, we introduce an extension of $\mathcal{ALC}$ to represent both defeasible constructs on complex concepts and contextual defeasible subsumption. In Section 4, we address the most important question from the standpoint of knowledge representation and reasoning with defeasible ontologies, namely that of entailment from defeasible knowledge bases. In particular, we present a semantic construction of contextual rational closure and provide a method for computing it. Finally, with Section 5 we conclude the paper.

The present paper is a revised and extended version of a paper presented at FoIKS 2018 [27]. Familiarity with the preferential approach to non-monotonic reasoning [41, 43, 47] will be helpful, as many of the intuitions and technical constructions are built on the propositional case. Whenever necessary, we refer the reader to the definitions and results in the relevant literature.

2 The description logic $\mathcal{ALC}$

The (concept) language of $\mathcal{ALC}$ is built upon a finite set of atomic *concept names* C, a finite set of *role names* R and a finite set of *individual names* I such that C, R and I are pairwise disjoint. With $A, B, \ldots$ we denote atomic concepts, with $r, s, \ldots$ role names, and with $a, b, \ldots$ individual names. Complex concepts are denoted by $C, D, \ldots$ and are built according to the following rule:

$$C ::= \top \mid \bot \mid \mathsf{C} \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.C \mid \forall r.C$$

With $\mathcal{L}_{\mathcal{ALC}}$ we denote the *language* of all $\mathcal{ALC}$ concepts.

The semantics of $\mathcal{L}_{\mathcal{ALC}}$ is the standard set-theoretic Tarskian semantics. An *interpretation* is a structure $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping concept names A to subsets $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, role names r to binary relations $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$, and individual names a to elements of the domain $\Delta^{\mathcal{I}}$, i.e., $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$.

Define $r^{\mathcal{I}}(x) := \{y \mid (x, y) \in r^{\mathcal{I}}\}$. We extend the interpretation function $\cdot^{\mathcal{I}}$ to also interpret complex concepts of $\mathcal{L}_{\mathcal{ALC}}$ in the following way:

$$\top^{\mathcal{I}} := \Delta^{\mathcal{I}};$$
$$\bot^{\mathcal{I}} := \emptyset;$$
$$(\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}};$$
$$(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}};$$
$$(C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}};$$
$$(\exists r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \cap C^{\mathcal{I}} \neq \emptyset\};$$
$$(\forall r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \subseteq C^{\mathcal{I}}\}.$$

Given $C, D \in \mathcal{L}_{\mathcal{ALC}}$, $C \sqsubseteq D$ is called a *subsumption statement*, or *general concept inclusion* (GCI), read "C is subsumed by D". $C \equiv D$ is an abbreviation for both $C \sqsubseteq D$ and $D \sqsubseteq C$. An $\mathcal{ALC}$ *TBox* $\mathcal{T}$ is a finite set of subsumption statements and formalises the *intensional* knowledge about a given domain of application. Given $C \in \mathcal{L}_{\mathcal{ALC}}$, $r \in \mathsf{R}$ and $a, b \in \mathsf{I}$, an *assertional statement* (*assertion*, for short) is an expression of the form $a : C$ or $(a, b) : r$. An $\mathcal{ALC}$ *ABox* $\mathcal{A}$ is a finite set of assertional statements formalising the *extensional* knowledge of the domain. We denote statements, both subsumption and assertional, with $\alpha, \beta, \ldots$. Given $\mathcal{T}$ and $\mathcal{A}$, with $\mathcal{KB} := \mathcal{T} \cup \mathcal{A}$ we denote an $\mathcal{ALC}$ *knowledge base*, a.k.a. an *ontology*.

An interpretation $\mathcal{I}$ *satisfies* a subsumption statement $C \sqsubseteq D$ (denoted $\mathcal{I} \Vdash C \sqsubseteq D$) if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. $\mathcal{I}$ *satisfies* an assertion $a : C$ (respectively, $(a, b) : r$), denoted $\mathcal{I} \Vdash a : C$ (respectively, $\mathcal{I} \Vdash (a, b) : r$), if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (respectively, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$).

An interpretation $\mathcal{I}$ is a *model* of a knowledge base $\mathcal{KB}$ (denoted $\mathcal{I} \Vdash \mathcal{KB}$) if $\mathcal{I} \Vdash \alpha$ for every $\alpha \in \mathcal{KB}$. A statement α is (classically) *entailed* by $\mathcal{KB}$, denoted $\mathcal{KB} \models \alpha$, if every model of $\mathcal{KB}$ satisfies α.

For more details on Description Logics in general and on $\mathcal{ALC}$ in particular, the reader is invited to consult the Description Logic Handbook [2].

3 Contextual defeasibility in DLs

In this section, we introduce an extension of $\mathcal{ALC}$ to represent both defeasible constructs on complex concepts and contextual defeasible subsumption. The logic presented here draws on the introduction of defeasible roles [21] and recent work on context-based defeasible subsumption [27].

3.1 Defeasible constructs

Our previous investigations of defeasible DLs included parameterised defeasible constructs on concepts based on preferential roles, in the form of defeasible value and existential restriction of the form $\mathbf{\forall} r.C$ and $\mathbf{\exists} r.C$. Intuitively, these concept descriptions refer respectively to individuals whose normal r-relationships are only to individuals from C, and individuals that have some normal r-relationship to an individual from C. However, while these constructs allowed for multiple preference orders on (the interpretation of) roles, only a single preference order on objects was assumed. This was somewhat of an anomaly, which we address here by adding context-based orderings on objects that are derived from preferential roles [23]. Briefly, each preferential role r, interpreted as a strict partial order on the binary product space of the domain, gives rise to a context-based order on objects as detailed in Definition 3 below.

The (concept) language of *defeasible* $\mathcal{ALC}$, or $d\mathcal{ALC}$, is built according to the following rule:

$$C ::= \top \mid \bot \mid \mathsf{C} \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \exists r.C \mid \forall r.C \mid \mathord{\reflectbox{E}}r.C \mid \forall\!\!\!\forall r.C$$

With $\mathcal{L}_{d\mathcal{ALC}}$ we denote the language of all $d\mathcal{ALC}$ concepts.

The extension of $\mathcal{ALC}$ we propose here also adds contextual defeasible subsumption statements to knowledge bases. Given $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and $r \in \mathsf{R}$, $C \sqsubseteq\!\!\!\sim_r D$ is a *defeasible subsumption statement* or *defeasible GCI*, read "C is usually subsumed by D in the context r". A $d\mathcal{ALC}$ *defeasible TBox* (or dTBox, for short) $\mathcal{D}$ is a finite set of defeasible GCIs. A $d\mathcal{ALC}$ *classical TBox* $\mathcal{T}$ (or TBox $\mathcal{T}$ for short) is a finite set of (classical) subsumption statements $C \sqsubseteq D$ (i.e., $\mathcal{T}$ may contain defeasible concept constructs, but not defeasible concept inclusions). As before, we shall use $\alpha, \beta, \ldots$ to denote statements, both classical and defeasible.

This begs the question of adding some version of "contextual classical subsumption" to the TBox, but, as we shall see in Section 3.2, this simply reduces to classical subsumption.

Given a $d\mathcal{ALC}$ classical TBox $\mathcal{T}$, an ABox $\mathcal{A}$ and a $d\mathcal{ALC}$ defeasible TBox $\mathcal{D}$, from now on we let $\mathcal{KB} := \mathcal{T} \cup \mathcal{D} \cup \mathcal{A}$ and refer to it as a $d\mathcal{ALC}$ *knowledge base* (alias defeasible ontology). Although we allow ABox assertions in the $d\mathcal{ALC}$ language and semantics, our focus in this paper, and in particular in Section 4, is on $d\mathcal{ALC}$ knowledge bases in which the ABox $\mathcal{A}$ is empty, i.e., $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$.

3.2 Preferential semantics

We anchor our semantic constructions in the well-known preferential approach to non-monotonic reasoning [41,43,47] and its extensions [9,10,8,12,25,26,20], especially those in DLs [19,21,35,45,48].

Let X be a set and let $<$ be a strict partial order on X. With $\min_< X := \{x \in X \mid \text{there is no } y \in X \text{ s.t. } y < x\}$ we denote the *minimal elements* of X w.r.t. $<$. With $\#X$ we denote the *cardinality* of X.

Definition 1 (Ordered interpretation) An *ordered interpretation* is a tuple $\mathcal{O} := \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$ such that:

- $\langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}} \rangle$ is an $\mathcal{ALC}$ interpretation, with $A^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}}$, for each $A \in \mathsf{C}$, $r^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$, for each $r \in \mathsf{R}$, and $a^{\mathcal{O}} \in \Delta^{\mathcal{O}}$, for each $a \in \mathsf{I}$, and
- $\ll^{\mathcal{O}} := \langle \ll_1^{\mathcal{O}}, \ldots, \ll_{\#\mathsf{R}}^{\mathcal{O}} \rangle$, where $\ll_i^{\mathcal{O}} \subseteq r_i^{\mathcal{O}} \times r_i^{\mathcal{O}}$, for $i = 1, \ldots, \#\mathsf{R}$, and such that each $\ll_i^{\mathcal{O}}$ is a well-founded strict partial order.

As an example, let $\mathsf{C} = \{\mathsf{Employee}, \mathsf{Company}, \mathsf{Student}, \mathsf{EmployedStudent}, \mathsf{Parent}, \mathsf{Tax}\}$, $\mathsf{R} = \{\mathsf{pays}, \mathsf{employedBy}, \mathsf{worksFor}\}$, and $\mathsf{I} = \{\mathsf{john}, \mathsf{ibm}, \mathsf{mary}\}$, with the respective obvious intuitions. Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$ be such that $\Delta^{\mathcal{O}} = \{x_i \mid 0 \leq i \leq 9\}$, and interpreting the above vocabulary as follows: $\mathsf{Employee}^{\mathcal{I}} = \{x_1, x_2, x_5, x_9\}$, $\mathsf{Company}^{\mathcal{I}} = \{x_6, x_{10}\}$, $\mathsf{Student}^{\mathcal{I}} = \{x_1, x_5, x_7, x_8\}$, $\mathsf{EmployedStudent}^{\mathcal{I}} = \{x_1, x_5\}$, $\mathsf{Parent}^{\mathcal{I}} = \{x_1, x_2, x_3\}$, $\mathsf{Tax}^{\mathcal{I}} = \{x_4\}$, $\mathsf{pays}^{\mathcal{I}} = \{(x_1, x_0), (x_5, x_4)\}$, $\mathsf{employedBy}^{\mathcal{I}} = \{(x_9, x_{10})\}$, $\mathsf{worksFor}^{\mathcal{I}} = \{(x_2, x_3), (x_2, x_6), (x_5, x_6), (x_9, x_6), (x_9, x_{10})\}$, $\mathsf{john}^{\mathcal{I}} = x_5$, $\mathsf{ibm}^{\mathcal{I}} = x_6$, $\mathsf{mary}^{\mathcal{I}} = x_2$.

Moreover, let us assume $\ll^{\mathcal{O}} = \langle \ll_{\mathsf{pays}}^{\mathcal{O}}, \ll_{\mathsf{employedBy}}^{\mathcal{O}}, \ll_{\mathsf{worksFor}}^{\mathcal{O}} \rangle$, where $\ll_{\mathsf{pays}}^{\mathcal{O}} = \{(x_5x_4, x_1x_0)\}$, $\ll_{\mathsf{employedBy}}^{\mathcal{O}} = \emptyset$, and $\ll_{\mathsf{worksFor}}^{\mathcal{O}} = \{(x_2x_6, x_2x_3), (x_9x_{10}, x_9x_6), (x_9x_6, x_5x_6), (x_9x_{10}, x_5x_6)\}$. (For the sake of readability, we shall henceforth sometimes write tuples of the form (x, y) as xy.)

Figure 1 below depicts the ordered interpretation $\mathcal{O}$ from the above example. In the picture, preference relations are represented by the dashed arrows. (Note the direction of the $\ll^{\mathcal{O}}$-arrows, which point from more preferred to less preferred pairs of objects. Also for the sake of readability, we omit the transitive $\ll^{\mathcal{O}}$-arrows.)

For example, we see that, informally, the individual john is a $\mathsf{Student}$ and an $\mathsf{Employee}$, ibm is a $\mathsf{Company}$, and john $\mathsf{worksFor}$ ibm. We also have unnamed objects in this domain, such as x_0 and x_1, where x_1 is a Parent but x_0 is not.

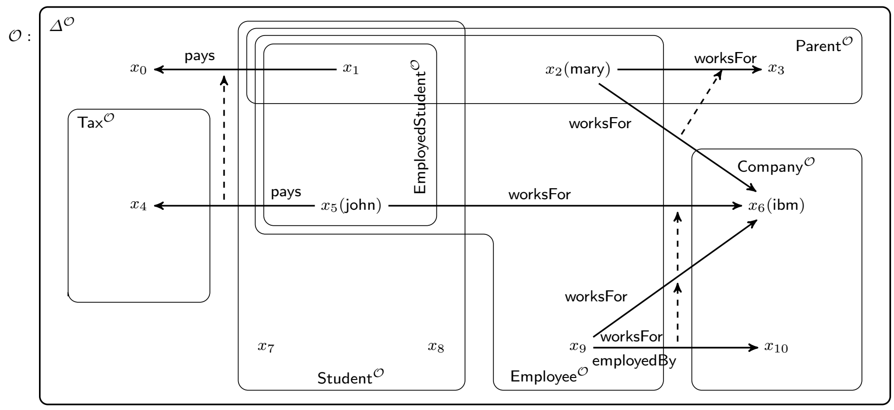

Fig. 1 A $d\mathcal{ALC}$ ordered interpretation.

Given $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, the intuition of $\Delta^{\mathcal{O}}$ and $\cdot^{\mathcal{O}}$ is the same as in a standard DL interpretation. The intuition underlying each of the orderings in $\ll^{\mathcal{O}}$ is that they play the role of *preference relations* (or *normality orderings*), in a sense similar to that introduced by Shoham [47] with a preference on worlds in a propositional setting and as investigated by Kraus et al. [41,43] and others [9,10,12,18,33]: the pairs (x, y) that are lower down in the ordering $\ll_i^{\mathcal{O}}$ are deemed as the most normal (or typical, or expected, or conventional) in the context of r_i.

The well-foundedness condition on each preference order assumed in Definition 1 will be used in the semantic construction of rational entailment of Section 4.2. It is sometimes replaced by the weaker smoothness condition in the literature on preferential reasoning [41], but this is not required here.

In the following definition we show how ordered interpretations can be extended to interpret the complex concepts of the language.

Definition 2 (Interpretation of concepts) Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, let $r \in \mathsf{R}$ and let $r^{\mathcal{O}|x} := r^{\mathcal{O}} \cap (\{x\} \times \Delta^{\mathcal{O}})$ (i.e., the restriction of the domain of $r^{\mathcal{O}}$ to $\{x\}$). The interpretation function $\cdot^{\mathcal{O}}$ interprets $d\mathcal{ALC}$ concepts as follows:

$$\top^{\mathcal{O}} := \Delta^{\mathcal{O}};$$
$$\bot^{\mathcal{O}} := \emptyset;$$
$$(\neg C)^{\mathcal{O}} := \Delta^{\mathcal{O}} \setminus C^{\mathcal{O}};$$
$$(C \sqcap D)^{\mathcal{O}} := C^{\mathcal{O}} \cap D^{\mathcal{O}};$$
$$(C \sqcup D)^{\mathcal{O}} := C^{\mathcal{O}} \cup D^{\mathcal{O}};$$
$$(\exists r.C)^{\mathcal{O}} := \{x \in \Delta^{\mathcal{O}} \mid r^{\mathcal{O}}(x) \cap C^{\mathcal{O}} \neq \emptyset\};$$
$$(\forall r.C)^{\mathcal{O}} := \{x \in \Delta^{\mathcal{O}} \mid r^{\mathcal{O}}(x) \subseteq C^{\mathcal{O}}\};$$
$$(\tilde{\exists} r.C)^{\mathcal{O}} := \{x \in \Delta^{\mathcal{O}} \mid \min_{\ll_r^{\mathcal{O}}}(r^{\mathcal{O}|x})(x) \cap C^{\mathcal{O}} \neq \emptyset\};$$
$$(\tilde{\forall} r.C)^{\mathcal{O}} := \{x \in \Delta^{\mathcal{O}} \mid \min_{\ll_r^{\mathcal{O}}}(r^{\mathcal{O}|x})(x) \subseteq C^{\mathcal{O}}\}.$$

If, as in Definition 2, the role name r is not indexed, we use r itself as subscript in $\ll_r^{\mathcal{O}}$. It is not hard to see that, analogously to the classical case, $\tilde{\forall}$ and $\tilde{\exists}$ are dual to each other.

In the ordered interpretation of Figure 1 we can see that, for example: $(\mathsf{Student} \sqcap \mathsf{Employee})^{\mathcal{O}} = \{x_1, x_5\}$; $(\exists \mathsf{worksFor}.\top \sqcap \forall \mathsf{worksFor}.\mathsf{Company})^{\mathcal{O}} = \{x_5, x_9\}$, and $(\exists \mathsf{worksFor}.\top \sqcap \tilde{\forall} \mathsf{worksFor}.\mathsf{Company})^{\mathcal{O}} = \{x_2, x_5, x_9\}$.

Definition 3 (Satisfaction) Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, $r \in \mathsf{R}$, $C, D \in \mathcal{L}_{d\mathcal{ALC}}$, and $a, b \in \mathsf{I}$. Define $\prec_r^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$ as follows:

$$\prec_r^{\mathcal{O}} := \{(x, y) \mid (\exists (x, z) \in r^{\mathcal{O}})(\forall (y, v) \in r^{\mathcal{O}})[((x, z), (y, v)) \in \ll_r^{\mathcal{O}}]\}.$$

The *satisfaction relation* $\Vdash$ is defined as follows:

$$\begin{array}{lll} \mathcal{O} \Vdash C \sqsubseteq D & \text{if} & C^{\mathcal{O}} \subseteq D^{\mathcal{O}}; \\ \mathcal{O} \Vdash C \precsim_r D & \text{if} & \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq D^{\mathcal{O}}; \\ \mathcal{O} \Vdash a : C & \text{if} & a^{\mathcal{O}} \in C^{\mathcal{O}}; \\ \mathcal{O} \Vdash (a,b) : r & \text{if} & (a^{\mathcal{O}}, b^{\mathcal{O}}) \in r^{\mathcal{O}}. \end{array}$$

If $\mathcal{O} \Vdash \alpha$, then we say $\mathcal{O}$ *satisfies* α. $\mathcal{O}$ satisfies a defeasible knowledge base $\mathcal{KB}$, written $\mathcal{O} \Vdash \mathcal{KB}$, if $\mathcal{O} \Vdash \alpha$ for every $\alpha \in \mathcal{KB}$, in which case we say $\mathcal{O}$ is a *model* of $\mathcal{KB}$. We say $C \in \mathcal{L}_{d\mathcal{ALC}}$ is *satisfiable* w.r.t. $\mathcal{KB}$ if there is a model $\mathcal{O}$ of $\mathcal{KB}$ s.t. $C^{\mathcal{O}} \neq \emptyset$. Likewise, $r \in \mathsf{R}$ is *satisfiable* w.r.t. $\mathcal{KB}$ if there is a model $\mathcal{O}$ of $\mathcal{KB}$ s.t. $r^{\mathcal{O}} \neq \emptyset$.

For example, in the ordered interpretation of Figure 1 we have that $\mathcal{O} \Vdash \mathsf{Employee} \precsim_{\mathsf{worksFor}} \neg\mathsf{Student}$. That is, employees are normally not students. We also have that $\mathcal{O} \Vdash \forall\mathsf{worksFor.Company} \sqsubseteq \mathbin{\forall\!\!\!\sim}\mathsf{worksFor.Company}$, but $\mathcal{O} \nVdash \mathbin{\forall\!\!\!\sim}\mathsf{worksFor.Company} \sqsubseteq \forall\mathsf{worksFor.Company}$. That is, every individual who only works for a company, normally only works for a company. But it is not the case that every individual who normally only works for a company, only works for a company.

The intuition of each context-based preference order $\prec_r^{\mathcal{O}}$ on objects in the domain of interpretation is to rank the relative normality of objects in the given context r. Different contexts can therefore in general give rise to different orderings. This does not imply that roles are identified with contexts, but rather that they provide a useful vehicle to relativise defeasible subsumption, and to introduce multiple preference orders on objects that are naturally read as providing context to preferences. It remains a modelling decision whether to peg a context to an existing role name, or introduce a new independent role name for this purpose.

Note that, for every $r \in \mathsf{R}$, $\precsim_r$ is *ampliative* and *non-monotonic*:

- Ampliativity: for every $\mathcal{O}$, if $\mathcal{O} \Vdash C \sqsubseteq D$, then $\mathcal{O} \Vdash C \precsim_r D$;
- Non-monotonicity: it is not generally the case that, for every $\mathcal{O}$, if $\mathcal{O} \Vdash C \precsim_r D$, then $\mathcal{O} \Vdash C \sqcap E \precsim_r D$ for every $E \in \mathcal{L}_{d\mathcal{ALC}}$.

It follows from Definition 3 that, if $\ll_r^{\mathcal{O}} = \emptyset$, i.e., if no r-tuple is preferred to another, then $\precsim_r$ reverts to $\sqsubseteq$. A similar observation holds for individual concept inclusions: if $(C \sqcap \exists r.\top)^{\mathcal{O}} = \emptyset$, then $C \precsim_r D$ reverts to $C \sqsubseteq D$. This reflects the intuition that the context r is taken into account through the preference order on $r^{\mathcal{O}}$. In the absence of any preference, the context becomes irrelevant. This also shows why the classical counterpart of $\precsim_r$ is independent of r — context is taken into account in the form of a preference order, but preference has no bearing on the semantics of $\sqsubseteq$.

Lemma 1 below shows that every ordered interpretation gives rise to a preference order on objects in the domain. Conversely, Lemma 2 shows that every strict partial order on objects in the domain $\Delta^{\mathcal{O}}$ can be obtained from some strict partial order on the interpretation of a new role name as in Definition 3. This means that the more traditional preference order on all objects in the domain, as usually adopted in the literature [18,19,33], is a special case of our proposal.

Lemma 1 *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, $r \in \mathsf{R}$ and let $\prec_r^{\mathcal{O}}$ be as in Definition 3. Then $\prec_r^{\mathcal{O}}$ is a well-founded strict partial order on $\Delta^{\mathcal{O}}$.*

Lemma 2 *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, and let $\prec$ be a well-founded strict partial order on $\Delta^{\mathcal{O}}$. Let $\mathcal{O}'$ be an extension of $\mathcal{O}$ with fresh role name $r' \in \mathsf{R}$ added, such that:*

$$\mathcal{O}' \Vdash \top \sqsubseteq \exists r'.\top;$$
$$\ll_{r'}^{\mathcal{O}'} := \{((x,z),(y,v)) \mid x \prec y \text{ and } (x,z),(y,v) \in r'^{\mathcal{O}'}\}.$$

Let $\prec_{r'}^{\mathcal{O}'}$ be as in Definition 3. Then $\prec = \prec_{r'}^{\mathcal{O}'}$.

Corollary 1 *Let $\mathcal{O}'$, $\prec$ and r' be as in Lemma 2, and let $\sqsubsetsim$ be defined by: $\mathcal{O}' \Vdash C \sqsubsetsim D$ if $\min_{\prec} C^{\mathcal{O}'} \subseteq D^{\mathcal{O}'}$. Then $\sqsubsetsim_{r'}$ has the same semantics as $\sqsubsetsim$.*

Corollary 1 states that, in the special case where the domain of a new designated context-providing role includes all objects, contextual defeasible subsumption coincides with defeasible subsumption based on a single preference order. In the more general parameterised case, consider for example the role $\mathsf{hasOrigin}$, which links individual wines to origins. Wine y is considered more exceptional than x w.r.t. its origin if it has some more exceptional origin link than x, and none that are less exceptional. This illustrates how context serves to inform the preference order on objects.

Contextual defeasible subsumption $\sqsubsetsim_r$ can therefore also be viewed as defeasible subsumption based on a preference order on objects in the domain of $r^{\mathcal{O}}$, bearing in mind that, in any given interpretation, it is dependent on $\ll_r^{\mathcal{O}}$. For the remainder of this paper, we use $\sqsubsetsim$ as abbreviation for $\sqsubsetsim_{r'}$, where r' is a new role name introduced as in Lemma 2.

This raises the question whether a preference order on objects in the range of $r^{\mathcal{O}}$ could be considered as an alternative. In a more expressive language allowing for role inverses, $\sqsubsetsim_{\mathsf{inv}(r)}$ achieves this goal [23], but in $d\mathcal{ALC}$, this would have to be added as an additional language construct.

The following result, of which the proof is analogous to that in the single-ordering case [15], shows that contextual defeasible subsumption is indeed an appropriate notion of non-monotonic subsumption:

Lemma 3 *For every $r \in \mathsf{R}$, $\sqsubsetsim_r$ is a preferential subsumption relation on concepts in that the following rules (a.k.a. the DL version of the well-known KLM-style postulates or properties) hold for every ordered interpretation $\mathcal{O}$, i.e., whenever $\mathcal{O}$ satisfies the antecedent, it satisfies the consequent as well:*

(Ref) $C \mathrel{\sqsubset\!\!\!\sim_r} C$ (LLE) $\dfrac{C \equiv D,\ C \mathrel{\sqsubset\!\!\!\sim_r} E}{D \mathrel{\sqsubset\!\!\!\sim_r} E}$ (And) $\dfrac{C \mathrel{\sqsubset\!\!\!\sim_r} D,\ C \mathrel{\sqsubset\!\!\!\sim_r} E}{C \mathrel{\sqsubset\!\!\!\sim_r} D \sqcap E}$

(Or) $\dfrac{C \mathrel{\sqsubset\!\!\!\sim_r} E,\ D \mathrel{\sqsubset\!\!\!\sim_r} E}{C \sqcup D \mathrel{\sqsubset\!\!\!\sim_r} E}$ (RW) $\dfrac{C \mathrel{\sqsubset\!\!\!\sim_r} D,\ D \sqsubseteq E}{C \mathrel{\sqsubset\!\!\!\sim_r} E}$ (CM) $\dfrac{C \mathrel{\sqsubset\!\!\!\sim_r} D,\ C \mathrel{\sqsubset\!\!\!\sim_r} E}{C \sqcap D \mathrel{\sqsubset\!\!\!\sim_r} E}$

The following property, which follows directly from the preferential semantics, makes explicit all classical information captured in the form of defeasible concept inclusions:

Lemma 4 *For every $r \in \mathsf{R}$, the following property holds for every ordered interpretation $\mathcal{O}$*

$$\text{(CC)}\ \frac{C \mathrel{\sqsubset\!\!\!\sim_r} \bot}{C \sqsubseteq \bot}$$

We now turn to a class of ordered interpretations that are of special importance in non-monotonic reasoning, namely *modular* interpretations. A strict partial order is called a *modular order* if its associated incomparability relation is transitive. That is, given a strict partial order $<$ over set X with converse relation $>$, the relation $\not< \cap \not>$ is a transitive relation. This condition partitions X into equivalence classes of incomparable elements. The strict partial order $<$ induces a linear order on these equivalence classes: Given two equivalence classes $[x]$ and $[y]$, we either have $z < u$ for all $z \in [x]$ and $u \in [y]$, or $z > u$ for all $z \in [x]$ and $u \in [y]$, or $[x] = [y]$.

Definition 4 (Modular interpretation) A *modular interpretation* is an ordered interpretation $\mathcal{O} := \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, where $\ll_r^{\mathcal{O}}$ is modular, for each $r \in \mathsf{R}$.

For example, in the modular interpretation depicted in Figure 2 below, the modular orders on $\mathsf{pays}^{\mathcal{O}}$, $\mathsf{worksFor}^{\mathcal{O}}$ and $\mathsf{employedBy}^{\mathcal{O}}$ induce the following respective equivalence classes: For the role pays, we have two equivalence classes $\{x_5x_4\}$ and $\{x_1x_0\}$; for $\mathsf{worksFor}$ we have $\{x_9x_{10}\}$, $\{x_9x_6, x_2x_6\}$ and $\{x_5x_6, x_2x_3\}$, and for $\mathsf{employedBy}$ the only equivalence class is $\{x_9x_{10}\}$.

Lemma 5 *Let $\mathcal{O}$ be a modular interpretation, $r \in \mathsf{R}$ and let $\prec_r^{\mathcal{O}}$ be as in Definition 3. Then $\prec_r^{\mathcal{O}}$ is a modular order.*

We call an ordered model of a knowledge base $\mathcal{KB}$ which is a modular interpretation a *modular model* of $\mathcal{KB}$. It turns out that if the preference order $\ll_r^{\mathcal{O}}$ on the interpretation of r is modular, then the defeasible subsumption $\mathrel{\sqsubset\!\!\!\sim_r}$ it induces is also *rational*:

Lemma 6 *For every $r \in \mathsf{R}$, $\mathrel{\sqsubset\!\!\!\sim_r}$ is a rational subsumption relation on concepts in that every modular interpretation $\mathcal{O}$ satisfies the following rational monotonicity property:*

$$\text{(RM)}\ \frac{C \mathrel{\sqsubset\!\!\!\sim_r} E,\ C \not\mathrel{\sqsubset\!\!\!\sim_r} \neg D}{C \sqcap D \mathrel{\sqsubset\!\!\!\sim_r} E}$$

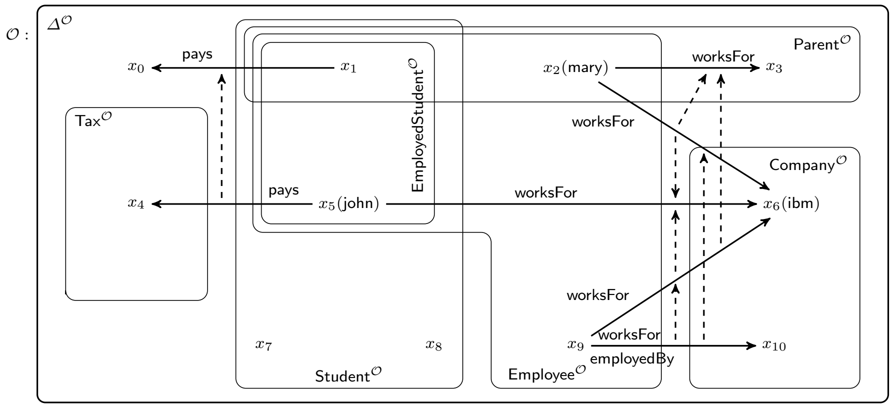

Fig. 2 A $d\mathcal{ALC}$ modular interpretation.

The importance of modularity therefore lies in that it delivers a rational defeasible subsumption relation.

We close this subsection by stating two lemmas that will be useful later on. Their proofs are generalisations of the respective results in the single-ordering case [14, 17], making use of the filtration construction for preferential Kripke models by Britz and Varzinczak [26], and we shall not provide them here.

Lemma 7 *If $\mathcal{KB}$ has a modular model, then it has a finite modular model.*

Lemma 8 *If $\mathcal{KB}$ has a modular model that is a counter-model to $C \mathrel{\sqsubset\!\!\!\sim}_r D$, then it has a finite modular model that is a counter-model to $C \mathrel{\sqsubset\!\!\!\sim}_r D$.*

3.3 Modelling with contexts

The motivation for defeasible knowledge bases is to represent defeasible knowledge, and to reason over defeasible ontologies. We now illustrate the different aspects of defeasibility that can be expressed in $d\mathcal{ALC}$. We first consider defeasible existential restriction:

$$\mathsf{Cheninblanc} \sqcap \exists\mathsf{hasAroma.Wood} \sqsubseteq \exists\mathsf{hasStyle.Wooded}$$

This statement can be read as: "Chenin blanc wines that normally have a wood aroma are wooded". That is, any Chenin blanc wine that has a characteristic wood aroma, has a wooded wine style. For an example of defeasible subsumption, consider the statement

$$\mathsf{Cheninblanc} \mathrel{\sqsubset\!\!\!\sim} \exists\mathsf{hasAroma.Floral}$$

where $\mathrel{\sqsubset\!\!\!\sim}$ is as in Corollary 1, which states that Chenin blanc wines usually have some floral aroma. That is, the most usual Chenin blanc wines all have some floral aroma. Similarly,

$$\mathsf{Cheninblanc} \mathrel{\sqsubseteq\!\!\!\sim} \forall \mathsf{hasOrigin}.\mathsf{Loire}$$

states that Chenin blanc wines usually come only from the Loire Valley. Now suppose we have a Chenin blanc wine x, which comes from the Loire Valley but does not have a floral aroma, and another Chenin blanc wine y which has a floral aroma but comes from Languedoc. No model of this ontology can simultaneously have $x \prec y$ w.r.t. origin and $y \prec x$ w.r.t. aroma. There can therefore be no model that accurately models reality.

This is precisely the limitation imposed by having only a single ordering on objects, as is broadly assumed by preferential approaches to defeasible DLs [18,19,33,35,37], and the motivation for introducing context-based defeasible subsumption. Although the two defeasible statements are not inconsistent, the presence of both rules out certain intended models. In contrast, with contextual defeasible subsumption, both subsumption statements can be expressed *and* x and y can have incompatible preferential relationships in the same model:

$$\mathsf{Cheninblanc} \mathrel{\sqsubseteq\!\!\!\sim}_{\mathsf{hasAroma}} \exists \mathsf{hasAroma}.\mathsf{Floral}$$

$$\mathsf{Cheninblanc} \mathrel{\sqsubseteq\!\!\!\sim}_{\mathsf{hasOrigin}} \forall \mathsf{hasOrigin}.\mathsf{Loire}$$

Note that these statements cannot be changed to:

$$\mathsf{Cheninblanc} \sqsubseteq \exists \mathsf{hasAroma}.\mathsf{Floral}$$

$$\mathsf{Cheninblanc} \sqsubseteq \forall \mathsf{hasOrigin}.\mathsf{Loire}$$

as the latter state that *every* Chenin blanc wine has some characteristic floral aroma and is usually exclusive to the Loire Valley. This rules out the possibility of a Chenin blanc without a floral aroma, or one that comes only (or just typically) from Languedoc.

We can also add subsumption statements indexed by different contextual roles. For example:

$$\mathsf{Cheninblanc} \mathrel{\sqsubseteq\!\!\!\sim} \exists \mathsf{hasAcidity}.(\mathsf{Medium} \sqcup \mathsf{High})$$

$$\mathsf{Cheninblanc} \mathrel{\sqsubseteq\!\!\!\sim}_{\mathsf{hasOrigin}} \exists \mathsf{hasAcidity}.\mathsf{High}$$

state that Chenin blanc wines usually have a medium or high acidity, whereas in the context of origin, Chenin blanc wines usually have a high acidity.

We conclude this section with an example from the access control domain. Let $\mathsf{C} := \{\mathsf{Intern}, \mathsf{Employee}, \mathsf{Graduate}, \mathsf{ResearchAssociate}, \mathsf{Classified}\}$, and $\mathsf{R} := \{\mathsf{access}, \mathsf{hasE}, \mathsf{hasQ}\}$. The intuition of the concept names are evident, while the intuition of the role names are respectively 'has access to', 'has employment status' and 'has qualification'. Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ with

$$\mathcal{T} = \left\{ \begin{array}{l} \mathsf{Intern} \sqsubseteq \mathsf{Employee}, \\ \mathsf{Employee} \sqsubseteq \exists \mathsf{hasE}.\top, \\ \mathsf{Graduate} \sqsubseteq \exists \mathsf{hasQ}.\top \end{array} \right\}$$

and

$$\mathcal{D} = \left\{ \begin{array}{l} \mathsf{Employee} \mathrel{\sqsubset\!\!\!\!\sim} \exists\mathsf{access.Classified}, \\ \mathsf{Intern} \mathrel{\sqsubset\!\!\!\!\sim} \neg\exists\mathsf{access.Classified}, \\ \mathsf{Intern} \sqcap \mathsf{Graduate} \mathrel{\sqsubset\!\!\!\!\sim} \exists\mathsf{access.Classified}, \\ \mathsf{ResearchAssociate} \mathrel{\sqsubset\!\!\!\!\sim} \neg\mathsf{Employee}, \\ \mathsf{ResearchAssociate} \mathrel{\sqsubset\!\!\!\!\sim} \mathsf{Graduate} \end{array} \right\}$$

Intuitively, $\mathcal{T}$ states that interns are employees, employees have some employment and graduates have some qualification. $\mathcal{D}$ states that employees usually have access to classified information, but interns don't, except for graduate interns. Also, research associates are usually not employees, and they are usually graduates.

It follows from Lemma 3 that the following statements are satisfied in any ordered model of $\mathcal{KB}$:

$$\mathsf{Intern} \sqcap \mathsf{Graduate} \mathrel{\sqsubset\!\!\!\!\sim} \neg\mathsf{Graduate} \sqcup \exists\mathsf{access.Classified} \quad \text{(RW)}$$

$$\mathsf{Intern} \sqcap \neg\mathsf{Graduate} \mathrel{\sqsubset\!\!\!\!\sim} \mathsf{Intern} \sqcap \neg\mathsf{Graduate} \quad \text{(Ref)}$$

$$\mathsf{Intern} \sqcap \neg\mathsf{Graduate} \mathrel{\sqsubset\!\!\!\!\sim} \neg\mathsf{Graduate} \sqcup \exists\mathsf{access.Classified} \quad \text{(RW)}$$

$$(\mathsf{Intern} \sqcap \mathsf{Graduate}) \sqcup (\mathsf{Intern} \sqcap \neg\mathsf{Graduate}) \mathrel{\sqsubset\!\!\!\!\sim} \neg\mathsf{Graduate} \sqcup \exists\mathsf{access.Classified} \quad \text{(Or)}$$

$$\mathsf{Intern} \mathrel{\sqsubset\!\!\!\!\sim} \neg\mathsf{Graduate} \sqcup \exists\mathsf{access.Classified} \quad \text{(LLE)}$$

$$\mathsf{Intern} \mathrel{\sqsubset\!\!\!\!\sim} (\neg\mathsf{Graduate} \sqcup \exists\mathsf{access.Classified}) \sqcap \neg\exists\mathsf{access.Classified} \quad \text{(And)}$$

$$\mathsf{Intern} \mathrel{\sqsubset\!\!\!\!\sim} \neg\mathsf{Graduate} \quad \text{(RW)}$$

One could similarly ask whether intern research associates are usually graduates, and whether they should usually have access to classified information in every ordered model of $\mathcal{KB}$. It soon becomes clear that modelling defeasible information is more challenging than modelling classical information, and that it becomes problematic when defeasible information relating to different contexts are not modelled independently.

Suppose, for example, that John is a graduate research associate who is also an employee, and Maria is a research associate who is neither a graduate nor an employee. In any ordered model of $\mathcal{KB}$, both John and Maria are exceptional in the class of research associates. This follows because John is an exceptional research associate w.r.t. employment status, and Maria is an exceptional research associate w.r.t. qualification. Also, in any ordered model of $\mathcal{KB}$ John and Maria are either incomparable, or one of them is more exceptional than the other. Because context has not been taken into account, there is no model in which John is more exceptional than Maria w.r.t. employment, but Maria is more exceptional than John w.r.t. qualification.

This reflects a modelling inaccuracy resulting from the adoption of a single preference order on objects. Using hasE and hasQ as contexts allows us to axiomatise defeasible statements relating to different contexts independently, and to avoid unintended interactions between defeasible axioms. For example:

$$\mathsf{ResearchAssociate} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Employee}$$

$$\mathsf{ResearchAssociate} \sqsubsetsim_{\mathsf{hasQ}} \mathsf{Graduate}$$

This raises the question whether allowing multiple contexts simply corresponds to a modelling intuition or whether it adds to the expressivity of the language. To answer this question, we first need to investigate the notion of *entailment* — determining what follows from a knowledge base.

4 Entailment in $d\mathcal{ALC}$

Given a $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$, we are interested in the reasoning task of *entailment of statements* from $\mathcal{KB}$. That is, given the knowledge specified in $\mathcal{KB}$, how do we decide what other subsumption statements follow from $\mathcal{KB}$?

As stated earlier, our focus in this paper is on $d\mathcal{ALC}$ knowledge bases in which the ABox $\mathcal{A}$ is empty. For the remainder of the present section, we therefore assume that $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$. This is done to simplify the technical presentation. Rational reasoning in the presence of an ABox has been investigated in the case of a single preference order on objects and the strategy followed there should in principle apply here, too [14,30]. In Section 4.1, we first introduce the natural generalisation of entailment to a preferential setting. Thereafter we consider the additional assumption of modularity on preferential models. This serves as motivation for our semantic characterisation of rational entailment in Section 4.2.

4.1 Preferential entailment

In order to get to a definition of entailment for $d\mathcal{ALC}$, an obvious starting point is to adopt a Tarskian notion thereof:

Definition 5 (Preferential entailment) A statement α is *preferentially entailed* by a $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\mathsf{pref}} \alpha$, if every ordered model of $\mathcal{KB}$ satisfies α.

Preferential entailment can be decided via the preferential tableau-based method we have defined in recent work [28].

We start by considering the appropriateness of such a notion of entailment.

Definition 6 ($\mathcal{KB}$-induced $\sqsubsetsim_r$) Let $\mathcal{KB}$ be a $d\mathcal{ALC}$ knowledge base and let $r \in \mathsf{R}$. With $\sqsubsetsim_r^{\mathcal{KB}} := \{C \sqsubsetsim_r D \mid \mathcal{KB} \models_{\mathsf{pref}} C \sqsubsetsim_r D\}$ we denote the *$\mathcal{KB}$-induced defeasible subsumption relation* in context r.

The result below follows immediately from Lemma 3 and the definition of preferential entailment above.

Corollary 2 *For every $r \in \mathsf{R}$ and every d$\mathcal{ALC}$ knowledge base $\mathcal{KB}$, $\sqsubset\!\!\!\sim_r^{\mathcal{KB}}$ is preferential, i.e., $\sqsubset\!\!\!\sim_r^{\mathcal{KB}}$ is closed under the rules in Lemma 3.*

In other words, Corollary 2 ensures that preferential entailment always delivers a set of subsumption statements satisfying the basic KLM properties for contextual non-monotonic reasoning.

Of course, preferential entailment is not always desirable, one of the reasons being that it is monotonic, courtesy of the Tarskian notion of consequence it relies on (see Definition 5). In most cases, as witnessed by the great deal of work in the non-monotonic reasoning community, a move towards rationality is in order. Thanks to the definitions above and the result in Lemma 6, we already know where to start looking for it:

Definition 7 (Modular entailment) A statement α is *modularly entailed* by a d$\mathcal{ALC}$ knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\mathsf{mod}} \alpha$, if every modular model of $\mathcal{KB}$ satisfies α.

It follows from Definitions 5 and 7 respectively that both preferential and modular entailment are *explosive* — if $\mathcal{KB}$ does not have an ordered (resp., modular) model, then it entails any d$\mathcal{ALC}$ statement. Defeasible subsumption provides a mechanism to resolve classical inconsistencies, but consistency remains a prerequisite for non-trivial reasoning.

Unfortunately, modular entailment falls short of providing us with an appropriate notion of non-monotonic entailment. This is so because it coincides with preferential entailment, as the following result, adapted from a well-known similar result in the propositional case [43, Theorem 4.2], shows.

Lemma 9 *For every d$\mathcal{ALC}$ knowledge base $\mathcal{KB}$, and every statement α, $\mathcal{KB} \models_{\mathsf{mod}} \alpha$ if and only if $\mathcal{KB} \models_{\mathsf{pref}} \alpha$.*

More fundamentally, this means the set of contextual $\sqsubset\!\!\!\sim$-statements induced by a knowledge base (cf. Definition 6) via modular entailment need not satisfy the rational monotonicity property. In what follows, we overcome precisely this defficiency.

4.2 Rational entailment

In this section, we introduce a definition of semantic entailment which is appropriate in the light of the discussion above. The constructions are for the most part inspired by the work by Booth and Paris [11] in the propositional case and those by Britz et al. [15,17] and Giordano et al. [37,38] in a single-ordered preferential DL setting. (We shall give a corresponding proof-theoretic characterisation of such a notion of entailment in Section 4.3.)

Let $\mathcal{KB}$ be a defeasible knowledge base and let Δ be a fixed countably infinite set. We define

$$Mod_{\Delta}(\mathcal{KB}) := \{\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle \mid \mathcal{O} \Vdash \mathcal{KB}, \mathcal{O} \text{ is modular and } \Delta^{\mathcal{O}} = \Delta\}.$$

The following result shows that the set $Mod_{\Delta}(\mathcal{KB})$ characterises modular entailment:

Lemma 10 *Let Δ be a fixed, countably infinite set. For every $\mathcal{KB}$, every $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $r \in \mathsf{R}$, $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubset\!\!\!\sim_r D$ if and only if $\mathcal{O} \Vdash C \sqsubset\!\!\!\sim_r D$, for every $\mathcal{O} \in Mod_{\Delta}(\mathcal{KB})$.*

Since Δ is countable, for every $\mathcal{O} \in Mod_{\Delta}(\mathcal{KB})$, we can partition $\Delta \times \Delta$ into a sequence of layers $(L_0, \ldots, L_n, \ldots)$, where, for each $i \geq 0$, $L_i := \langle L_i^{r_1}, \ldots, L_i^{r_{\#\mathsf{R}}} \rangle$, and such that, for every $x, y \in \Delta$ and every $r \in \mathsf{R}$, $(x, y) \in L_0^r$ iff $(x, y) \in \min_{\ll_r^{\mathcal{O}}} r^{\mathcal{O}}$ and $(x, y) \in L_{i+1}^r$ iff $(x, y) \in \min_{\ll_r^{\mathcal{O}}} (r^{\mathcal{O}} \setminus \bigcup_{0 \leq j \leq i} L_j^r)$. (That these constructions are well defined follows from the fact that, for every $r \in \mathsf{R}$, $\ll_r^{\mathcal{O}}$ is modular and well-founded.)

Definition 8 (Height of a pair) Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$ be a modular interpretation over a countable domain $\Delta^{\mathcal{O}}$, let $x, y \in \Delta^{\mathcal{O}}$ and let $r \in \mathsf{R}$. The *height* of (x, y) in $\mathcal{O}$ w.r.t. r is denoted $h_{\mathcal{O}}(x, y, r)$ and is defined by: $h_{\mathcal{O}}(x, y, r) := i$ if $(x, y) \in L_i^r$.

Intuitively, the lower the height of (x, y) w.r.t. r in an interpretation $\mathcal{O}$, the more typical (or normal, or conventional) (x, y) is.

For example, in the modular interpretation $\mathcal{O}$ depicted in Figure 2 the height of the respective pairs in $\mathcal{O}$ w.r.t. pays, worksFor and employedBy are as follows: $h_{\mathcal{O}}(x_5, x_4, \mathsf{pays}) = 0$; $h_{\mathcal{O}}(x_1, x_{10}, \mathsf{pays}) = 1$; $h_{\mathcal{O}}(x_9, x_{10}, \mathsf{worksFor}) = 0$; $h_{\mathcal{O}}(x_9, x_6, \mathsf{worksFor}) = h_{\mathcal{O}}(x_2, x_6, \mathsf{worksFor}) = 1$; $h_{\mathcal{O}}(x_5, x_6, \mathsf{worksFor}) = h_{\mathcal{O}}(x_2, x_3, \mathsf{worksFor}) = 2$, and $h_{\mathcal{O}}(x_9, x_{10}, \mathsf{employedBy}) = 0$.

For every $\mathcal{O} \in Mod_{\Delta}(\mathcal{KB})$, given the induced $\prec_r^{\mathcal{O}}$ for each $r \in \mathcal{R}$ (Definition 3) and thanks to Lemma 5, Δ too can be partitioned into a sequence of (multiple) layers of *objects* of the domain. This allows us to define the height of an object.

Definition 9 (Height of an object) Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$ be a modular interpretation over a countable domain $\Delta^{\mathcal{O}}$, let $x \in \Delta^{\mathcal{O}}$ and let $r \in \mathsf{R}$. The *height* of x in $\mathcal{O}$ w.r.t. r is denoted $h_{\mathcal{O}}(x, r)$ and is defined by:

$$h_{\mathcal{O}}(x, r) := \begin{cases} \min\{h_{\mathcal{O}}(x, y, r) \mid (x, y) \in r^{\mathcal{O}}\}, & \text{if } x \text{ is in the domain of } r^{\mathcal{O}}; \\ \omega \text{ otherwise.} \end{cases}$$

Intuitively, the lower the height of an object in an interpretation $\mathcal{O}$, the more typical (or normal, or conventional) the object is in $\mathcal{O}$ in the context r. We can also think of a level of typicality for concepts: the *height of a concept* $C \in \mathcal{L}_{d\mathcal{ALC}}$ in $\mathcal{O}$ in the context r is the index of the layer to which the restriction of the concept's extension to its $\prec_r^{\mathcal{O}}$-minimal elements belong, i.e., $h_{\mathcal{O}}(C, r) = i$ if $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \neq \emptyset$ and $h_{\mathcal{O}}(x, r) = i$ for every $x \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$. Since $\mathcal{O}$ is a modular interpretation, all such $\prec_r^{\mathcal{O}}$-minimal elements are guaranteed to have the same height (i.e., they fall into the same equivalence class of incomparable elements). If x is not in the domain of $r^{\mathcal{O}}$, its height is defined as ω, the smallest ordinal greater than all natural numbers.

For example, we can derive from the heights of pairs in the modular interpretation of Figure 2 that $h_{\mathcal{O}}(x_9, \mathsf{worksFor}) = 0$, $h_{\mathcal{O}}(x_2, \mathsf{worksFor}) = 1$ and $h_{\mathcal{O}}(x_5, \mathsf{worksFor}) = 2$. All other objects have height ω w.r.t. $\mathsf{worksFor}$.

We can now use the set $Mod_{\Delta}(\mathcal{KB})$ as a springboard to introduce what will turn out to be a canonical modular interpretation for $\mathcal{KB}$.

Definition 10 (Big modular interpretation) Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base and define $\mathcal{O}_{\oplus}^{\mathcal{KB}} := \langle \Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}, \cdot^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}, \ll^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \rangle$, where

- $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} := \coprod_{\mathcal{O} \in Mod_{\Delta}(\mathcal{KB})} \Delta^{\mathcal{O}}$, i.e., the disjoint union of the domains from $Mod_{\Delta}(\mathcal{KB})$, where each $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle \in Mod_{\Delta}(\mathcal{KB})$ has the elements $x, y, \ldots$ of its domain renamed as $x_{\mathcal{O}}$, $y_{\mathcal{O}}$, $\ldots$ so that they are all distinct in $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$;
- $x_{\mathcal{O}} \in A^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ if $x \in A^{\mathcal{O}}$;
- $(x_{\mathcal{O}}, y_{\mathcal{O}'}) \in r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ if $\mathcal{O} = \mathcal{O}'$ and $(x, y) \in r^{\mathcal{O}}$;
- $(x_{\mathcal{O}}, y_{\mathcal{O}'}) \ll_r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} (x'_{\mathcal{O}}, y'_{\mathcal{O}'})$ if $h_{\mathcal{O}}(x, y, r) < h_{\mathcal{O}'}(x', y', r)$.

Intuitively, a big modular interpretation consists in a canonical modular interpretation having as domain the disjoint union of the domains of all modular interpretations of a knowledge base with domain Δ. It preserves the respective interpretations of concept names and role names w.r.t. each interpretation in the disjoint union. Moreover, it defines preferences on the elements of role interpretations that are faithful to their relative heights in the respective modular interpretations used in its construction.

The following result establishes that modular interpretations are closed under disjoint union.

Lemma 11 *If $\mathcal{KB}$ is modularly satisfiable, then $\mathcal{O}_{\oplus}^{\mathcal{KB}}$ is a modular interpretation.*

The proofs for the two lemmas below follow from the definition of $\mathcal{O}_{\oplus}^{\mathcal{KB}}$:

Lemma 12 *For every $C \in \mathcal{L}_{d\mathcal{ALC}}$ and $\mathcal{O} \in Mod_{\Delta}(\mathcal{KB})$, $x_{\mathcal{O}} \in C^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ if and only if $x \in C^{\mathcal{O}}$.*

Lemma 13 *For every $r \in \mathsf{R}$ and $\mathcal{O} \in Mod_{\Delta}(\mathcal{KB})$, and for every $(x, y) \in r^{\mathcal{O}}$, $h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(x_{\mathcal{O}}, y_{\mathcal{O}}, r) = h_{\mathcal{O}}(x, y, r)$.*

The three results above allow us to show the following:

Corollary 3 *If $\mathcal{KB}$ is modularly satisfiable, then $\mathcal{O}_{\oplus}^{\mathcal{KB}}$ is a modular model of $\mathcal{KB}$.*

Given $\mathcal{O}_{\oplus}^{\mathcal{KB}}$, we can then define contextual modular orderings $\prec_r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ on the domain $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ in the same way as in Definition 3. Thanks to Lemma 5, each of these orderings is a modular order on $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$.

Armed with the definitions and results above, we are now ready to provide an alternative definition of entailment in $d\mathcal{ALC}$:

Definition 11 (Rational entailment) A statement α is *rationally entailed* by a knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\mathsf{rat}} \alpha$, if $\mathcal{O}_{\oplus}^{\mathcal{KB}} \Vdash \alpha$ whenever $\mathcal{O}_{\oplus}^{\mathcal{KB}}$ is defined, or if $\mathcal{KB}$ is modularly unsatisfiable.

Corollary 4 *Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base. $\mathcal{KB} \models_{\mathsf{rat}} C \sqsubseteq\!\!\!\sim_r D$ if and only if $h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(C \sqcap D, r) < h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(C \sqcap \neg D, r)$ or $h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(C \sqcap \neg D, r') = \omega$, where r' is a new role name introduced as in Lemma 2.*

The result below follows from the fact that the big modular interpretation is a modular interpretation of the knowledge base (Corollary 3) together with Lemma 6.

Corollary 5 *Let $\mathcal{KB}$ be a defeasible knowledge base. For every $r \in \mathsf{R}$, $\{C \sqsubseteq\!\!\!\sim_r D \mid \mathcal{KB} \models_{\mathsf{rat}} C \sqsubseteq\!\!\!\sim_r D\}$ is rational, i.e., it is closed under the preferential rules of Lemma 3 as well as the RM rule of Lemma 6.*

In conclusion, $\models_{\mathsf{rat}}$ is the notion of entailment for contextual defeasible subsumption we were looking for. Modular entailment falls short of what is needed, since it does not in general deliver a rational version of contextual defeasible subsumption, whereas Corollary 5 shows that rational entailment does.

4.3 Computing contextual rational closure

In the remainder of the section, we discuss a known instance of entailment for defeasible reasoning that meets all the requirements of rational entailment. It is a generalisation of the DL version of the propositional *rational closure* studied by Lehmann and Magidor [43], to deal with context-based rational defeasible entailment. We present a proof-theoretic characterisation here, based on the work of Casini and Straccia [31,32]; an alternative semantic characterisation of rational closure in DLs (without contexts) was proposed by Giordano and others [37,38].

Rational closure is a form of inferential closure based on modular entailment $\models_{\mathsf{mod}}$, but it extends its inferential power. Such an extension of modular entailment is obtained formalising what is called the principle of *presumption of typicality* [42, Section 3.1]. That is, we always assume that we are dealing with the most typical possible situation compatible with the information at our disposal. The main result of this section is the correspondence result of Theorem 1, which relates rational closure to rational entailment introduced in Definition 11 above. We first define what it means for a concept to be *exceptional* in a given context:

Definition 12 (Contextual exceptionality) A concept C is *exceptional in the context* r in the defeasible knowledge base $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ if $\mathcal{KB} \models_{\mathsf{mod}} \top \sqsubseteq\!\!\!\sim_r \neg C$. A defeasible subsumption statement $C \sqsubseteq\!\!\!\sim_r D$ is exceptional in the context r in $\mathcal{KB}$ if C is exceptional in the context r in $\mathcal{KB}$.

So, a concept C is considered exceptional in a given context in a knowledge base if it is not possible to have a modular model of the knowledge base in which there is a typical individual (i.e., an individual at least as typical as all the others in the domain of interpretation) that is an instance of the concept C. Applying the notion of exceptionality iteratively, we associate with every concept C and context r a *rank* in the knowledge base $\mathcal{KB}$, which we denote by $rank_{\mathcal{KB}}(C, r)$. We extend this to subsumption statements, and associate with every context r and contextual defeasible concept inclusion $C \sqsubseteq\!\!\!\sim_r D$ a rank, denoted $rank_{\mathcal{KB}}(C \sqsubseteq\!\!\!\sim_r D, r)$ and abbreviated as $rank_{\mathcal{KB}}(C \sqsubseteq\!\!\!\sim_r D)$:

1. Let $rank_{\mathcal{KB}}(C, r) = 0$ if C is not exceptional in the context of r and $\mathcal{KB}$, and let $rank_{\mathcal{KB}}(C \sqsubseteq\!\!\!\sim_r D) = 0$ for every defeasible statement having C as antecedent, with $rank_{\mathcal{KB}}(C, r) = 0$. The set of statements in $\mathcal{D}$ with rank 0 is denoted as $\mathcal{D}_0^{rank}$.
2. Let $rank_{\mathcal{KB}}(C, r) = 1$ if C does not have a rank of 0 in the context of r and it is not exceptional in the knowledge base $\mathcal{KB}^1$ composed of $\mathcal{T}$ and the exceptional part of $\mathcal{D}$, that is, $\mathcal{KB}^1 := \langle \mathcal{T}, \mathcal{D} \setminus \mathcal{D}_0^{rank} \rangle$. If $rank_{\mathcal{KB}}(C, r) = 1$, then let $rank_{\mathcal{KB}}(C \sqsubseteq\!\!\!\sim_r D) = 1$ for every statement $C \sqsubseteq\!\!\!\sim_r D$. The set of statements in $\mathcal{D}$ with rank 1 is denoted $\mathcal{D}_1^{rank}$.
3. In general, for $i > 0$, a tuple $\langle C, r \rangle$ is assigned a rank of i if it does not have a rank of $i - 1$ and it is not exceptional in $\mathcal{KB}^i := \langle \mathcal{T}, \mathcal{D} \setminus \bigcup_{j=0}^{i-1} \mathcal{D}_j^{rank} \rangle$. If $rank_{\mathcal{KB}}(C, r) = i$, then $rank_{\mathcal{KB}}(C \sqsubseteq\!\!\!\sim_r D) = i$ for every statement $C \sqsubseteq\!\!\!\sim_r D$. The set of statements in $\mathcal{D}$ with rank i is denoted $\mathcal{D}_i^{rank}$.
4. By iterating the previous steps, we eventually reach a subset $\mathcal{E} \subseteq \mathcal{D}$ such that all the statements in $\mathcal{E}$ are exceptional (since $\mathcal{D}$ is finite, we must reach such a point). We define the rank of the statements in $\mathcal{E}$ as ω, and the set $\mathcal{E}$ is denoted $\mathcal{D}_\omega^{rank}$.

Following on the procedure above, $\mathcal{D}$ is partitioned into a finite sequence $\langle \mathcal{D}_0^{rank}, \ldots, \mathcal{D}_n^{rank}, \mathcal{D}_\omega^{rank} \rangle$ $(n \geq 0)$, where $\mathcal{D}_\omega^{rank}$ may possibly be empty. So, through this procedure we can assign a rank to every context-based defeasible subsumption statement.

The application of the ranking procedure will be illustrated in the example presented in Section 4.4, but first, we present the main correspondence result of this section in Theorem 1 below.

The following result establishes that the ranking procedure correctly captures the semantic representation in the model $\mathcal{O}_{\oplus}^{\mathcal{KB}}$:

Lemma 14 *Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base, let $C \in \mathcal{L}_{d\mathcal{ALC}}$, let $r \in \mathsf{R}$ be satisfiable, and let $i \leq \omega$. If $rank_{\mathcal{KB}}(C, r) = i$, then $h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(C, r) = i$.*

For a concept C to have a rank of ω in the context r corresponds to it not being satisfiable in the domain of r in any model of $\mathcal{KB}$, i.e., $\mathcal{KB} \models_{\mathsf{mod}} C \sqcap \exists r.\top \sqsubseteq \bot$. This is stated in Corollary 6 below, which follows from Lemma 14. A special case of the corollary is obtained when $\mathcal{KB} \models_{\mathsf{mod}} \top \sqsubseteq \exists r.\top$, as is the case in Lemma 2 and Corollary 1. This is made explicit by the property

(CC), which adds the classical ramifications of incoherencies that follow from defeasible statements in the defeasible TBox, to the TBox.

Corollary 6 *Let $C \in \mathcal{L}_{d\mathcal{ALC}}$ and let $r \in \mathsf{R}$ be satisfiable. Then $rank_{\mathcal{KB}}(C, r) = \omega$ if and only if $\mathcal{KB} \models_{\mathsf{mod}} C \sqcap \exists r.\top \sqsubseteq \bot$.*

Given $C \in \mathcal{L}_{d\mathcal{ALC}}$, we abbreviate the rank $rank_{\mathcal{KB}}(C, r')$ of C in the context provided by a new role name r' introduced as in Lemma 2, by $rank_{\mathcal{KB}}(C)$. Adapting Lehmann and Magidor's construction for propositional logic [43], the contextual rational closure of a knowledge base $\mathcal{KB}$ is now defined as follows:

Definition 13 (Contextual rational closure) Let $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and let $r \in \mathsf{R}$. Then $C \sqsubseteq\!\!\!\sim_r D$ is in the *rational closure* of a defeasible knowledge base $\mathcal{KB}$ if

$$rank_{\mathcal{KB}}(C \sqcap D, r) < rank_{\mathcal{KB}}(C \sqcap \neg D, r) \quad \text{or} \quad rank_{\mathcal{KB}}(C \sqcap \neg D) = \omega.$$

Informally, the above definition says that $C \sqsubseteq\!\!\!\sim_r D$ is in the rational closure of $\mathcal{KB}$ if the modular models of the knowledge base tell us that, in the context of r, some instances of $C \sqcap D$ are more plausible than all instances of $C \sqcap \neg D$. If r is satisfiable and C is disjoint from the domain of r in all modular models of $\mathcal{KB}$ (cf. Corollary 6), $C \sqsubseteq\!\!\!\sim_r D$ reverts to $C \sqsubseteq D$ and is evaluated as such. If r is unsatisfiable, the same reduction follows from the observation that all objects are incomparable w.r.t. r in any modular model of $\mathcal{KB}$.

Theorem 1 *Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base. For every $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $r \in \mathsf{R}$, $C \sqsubseteq\!\!\!\sim_r D$ is in the rational closure of $\mathcal{KB}$ if and only if $\mathcal{KB} \models_{\mathsf{rat}} C \sqsubseteq\!\!\!\sim_r D$.*

4.4 Rational reasoning with context in $d\mathcal{ALC}$ ontologies

The following example shows how ranks are assigned to concepts in a defeasible TBox, and used to determine rational entailment. We first consider only a single context $\mathsf{hasE} \in \mathsf{R}$ with intuition 'has employment', and then extend the example to demonstrate the strength of reasoning with multiple contexts.

Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ with $\mathcal{T} = \{\mathsf{Intern} \sqsubseteq \mathsf{Employee},\ \mathsf{Employee} \sqsubseteq \exists \mathsf{hasE}.\top\}$ and

$$\mathcal{D} = \left\{ \begin{array}{l} \mathsf{Employee} \sqsubseteq\!\!\!\sim_{\mathsf{hasE}} \exists \mathsf{access.Classified}, \\ \mathsf{Intern} \sqsubseteq\!\!\!\sim_{\mathsf{hasE}} \neg \exists \mathsf{access.Classified}, \\ \mathsf{Intern} \sqcap \mathsf{Graduate} \sqsubseteq\!\!\!\sim_{\mathsf{hasE}} \exists \mathsf{access.Classified} \end{array} \right\}$$

Examining the concepts on the LHS of each subsumption in $\mathcal{D}$, we get that:

1. $rank_{\mathcal{KB}}(\mathsf{Employee}, \mathsf{hasE}) = 0$, since it is possible to instantiate $\mathsf{Employee}$ with a typical individual. Hence it follows from Definition 12 that $\mathsf{Employee}$ is not exceptional in $\mathcal{KB}$.

2. $rank_{\mathcal{KB}}(\mathsf{Intern}, \mathsf{hasE}) \neq 0$ and $rank_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) \neq 0$, since it is not possible to instantiate either with a typical individual. That is, $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim_{\mathsf{hasE}} \neg\mathsf{Intern}$ and $\mathcal{KB} \models_{\mathsf{mod}} \top \precsim_{\mathsf{hasE}} \neg(\mathsf{Intern} \sqcap \mathsf{Graduate})$. It therefore follows from Definition 12 that both concepts are exceptional in $\mathcal{KB}$.
3. $\mathcal{KB}^1$ is composed of $\mathcal{T}$ and $\mathcal{D} \setminus \mathcal{D}_0^{rank}$, which consists of the defeasible subsumptions in $\mathcal{D}$ except for $\mathsf{Employee} \precsim_{\mathsf{hasE}} \exists\mathsf{access.Classified}$;
4. $rank_{\mathcal{KB}}(\mathsf{Intern}, \mathsf{hasE}) = 1$, since Intern is not exceptional in $\mathcal{KB}^1$;
5. $rank_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) \neq 1$, since $\mathsf{Intern} \sqcap \mathsf{Graduate}$ is exceptional in $\mathcal{KB}^1$;
6. $\mathcal{KB}^2$ is composed of $\mathcal{T}$ and $\{\mathsf{Intern} \sqcap \mathsf{Graduate} \precsim_{\mathsf{hasE}} \exists\mathsf{access.Classified}\}$;
7. $\mathsf{Intern} \sqcap \mathsf{Graduate}$ is not exceptional in $\mathcal{KB}^2$ and therefore $rank_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) = 2$.

There are algorithms to compute rational closure [29,32,38,44] that can be adapted to account for context, but one can also apply Definition 13 to determine rational entailment. For example, since $rank_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) = 2$ and $rank_{\mathcal{KB}}(\mathsf{Intern} \sqcap \neg\mathsf{Graduate}, \mathsf{hasE}) = 1$, we find that interns are usually not graduates: $\mathcal{KB} \models_{\mathsf{rat}} \mathsf{Intern} \precsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$.

The context hasE is used to indicate that it is an individual's typicality in the context of employment which is under consideration. Now suppose that $\mathcal{KB}$ in the above example is extended to $\mathcal{KB}' = \mathcal{T}' \cup \mathcal{D}'$, where $\mathcal{T}' = \{\mathsf{Intern} \sqsubseteq \mathsf{Employee},\ \mathsf{Employee} \sqsubseteq \exists\mathsf{hasE}.\top,\ \mathsf{Graduate} \sqsubseteq \exists\mathsf{hasQ}.\top\}$ and

$$\mathcal{D}' = \left\{ \begin{array}{l} \mathsf{Employee} \precsim_{\mathsf{hasE}} \exists\mathsf{access.Classified}, \\ \mathsf{Intern} \precsim_{\mathsf{hasE}} \neg\exists\mathsf{access.Classified}, \\ \mathsf{Intern} \sqcap \mathsf{Graduate} \precsim_{\mathsf{hasE}} \exists\mathsf{access.Classified}, \\ \mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \neg\mathsf{Employee}, \\ \mathsf{ResearchAssociate} \precsim_{\mathsf{hasQ}} \mathsf{Graduate} \end{array} \right\}$$

The context hasQ is used here to indicate that it is an individual's typicality w.r.t. qualification which is under consideration. In our example, the rankings calculated above are not affected by the additional information, and can be recalculated as above. In addition, $rank_{\mathcal{KB}'}(\mathsf{ResearchAssociate}, \mathsf{hasE}) = 0$ and $rank_{\mathcal{KB}'}(\mathsf{ResearchAssociate}, \mathsf{hasQ}) = 0$, since ResearchAssociate can be instantiated with a typical individual relative to both contexts. It now follows that:

- In the context hasQ, interns who are also research associates are usually graduates: $\mathcal{KB}' \models_{\mathsf{rat}} \mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \precsim_{\mathsf{hasQ}} \mathsf{Graduate}$. This follows because $rank_{\mathcal{KB}'}(\mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasQ}) = 0$, whereas $rank_{\mathcal{KB}'}(\mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \sqcap \neg\mathsf{Graduate}, \mathsf{hasQ}) = 1$.
- In the context hasE, interns who are also research associates are usually not graduates: $\mathcal{KB}' \models_{\mathsf{rat}} \mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \precsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$. This follows because $rank_{\mathcal{KB}'}(\mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) = 2$, whereas $rank_{\mathcal{KB}'}(\mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \sqcap \neg\mathsf{Graduate}, \mathsf{hasE}) = 1$.

On the other hand, suppose we were restricted to a single context hasE, i.e., let $\mathcal{KB}'' = \mathcal{T}'' \cup \mathcal{D}''$, where $\mathcal{T}'' = \mathcal{T}$ and

$$\mathcal{D}'' = \left\{ \begin{array}{l} \mathsf{Employee} \precsim_{\mathsf{hasE}} \exists \mathsf{access}.\mathsf{Classified}, \\ \mathsf{Intern} \precsim_{\mathsf{hasE}} \neg\exists \mathsf{access}.\mathsf{Classified}, \\ \mathsf{Intern} \sqcap \mathsf{Graduate} \precsim_{\mathsf{hasE}} \exists \mathsf{access}.\mathsf{Classified}, \\ \mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \neg\mathsf{Employee}, \\ \mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \mathsf{Graduate} \end{array} \right\}$$

We then only get that $\mathcal{KB}'' \models_{\mathsf{rat}} \mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \precsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$.

Which one of these rational entailments is more intuitively correct can perhaps be understood better by looking at the postulates for non-monotonic reasoning in Lemmas 3 and 6, which provide a more intuitive perspective and insight than calculating rankings. Looking at models of $\mathcal{KB}'$, in particular $\mathcal{O}_{\oplus}^{\mathcal{KB}'}$, it follows from (RM) that $\mathcal{KB}' \models_{\mathsf{rat}} \mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \precsim_{\mathsf{hasQ}} \mathsf{Graduate}$. That is, in the context of qualification, since research associates are usually graduates, so are intern research associates. Also in $\mathcal{KB}'$, applying (RM) to $\mathsf{Intern} \precsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$ we get $\mathsf{Intern} \sqcap \mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$. That is, in the context of employment, since interns are usually not graduates, neither are intern research associates. Note also that neither of these entailments is warranted by modular entailment.

In contrast, in models of $\mathcal{KB}''$, including $\mathcal{O}_{\oplus}^{\mathcal{KB}''}$, the former deduction is blocked: we can apply (RW) to $\mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \neg\mathsf{Employee}$ to obtain $\mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \neg\mathsf{Intern}$. The application of (RM) is now blocked by the axiom $\mathsf{ResearchAssociate} \precsim_{\mathsf{hasE}} \neg\mathsf{Intern}$, hence we cannot conclude that $\mathcal{KB}'' \models_{\mathsf{rat}} \mathsf{ResearchAssociate} \sqcap \mathsf{Intern} \precsim_{\mathsf{hasE}} \mathsf{Graduate}$.

This example illustrates that even subtle modelling changes can have unexpected effects. Reasoning with contexts can add to the challenges of ontology modelling, and requires clear design principles.

5 Concluding remarks

In this paper, we have made a case for a parameterised notion of defeasible concept inclusion in description logics. We have shown that preferential roles can be used to take context into account, and to deliver a simple, yet powerful, notion of contextual defeasible subsumption. Technically, this addresses an important limitation in previous defeasible extensions of description logics, namely the restriction in the semantics of defeasible concept inclusion to a single preference order on objects. Semantically, it answers the question of the meaning of multiple preference orders, namely that they reflect different contexts.

We have presented context as an explanation of the intuition underlying the introduction of multiple preference orders on objects, with defeasibility introducing a new facet of contextual reasoning not present in deductive reasoning. This offers a semantic treatment of contextual defeasible subsumption which requires no extended vocabulary or further extension of the concept language, yet is informed by the semantic constraints resulting from the use of role names in the concept language. In contrast, an account of *deductive* reasoning with contexts in knowledge representation is not intrinsically linked to defeasible reasoning. The integration of defeasible description logics with such an account of contextual knowledge representation in description logics, for example, (possibly non-monotonic) contextualized knowledge repositories [13, 39] or two-sorted description logics of context [40], is orthogonal to our work, and has not yet been attempted.

Building on previous work in the KLM tradition, we have shown that restricting the preferential semantics to a modular semantics allows us to define the notion of rational entailment from a defeasible knowledge base, and to compute the rational closure of a knowledge base as an instance of rational entailment. Future work should consider the implementation of contextual rational closure, as well as the addition of an ABox. Much work is also required on the modelling side once a stable implementation exists. Contextual subsumption provides the user with more flexibility in making defeasible statements in ontologies, but comprehensive case studies are required to evaluate the approach.

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A Proofs

Lemma 1 *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, $r \in \mathsf{R}$ and let $\prec_r^{\mathcal{O}}$ be as in Definition 3. Then $\prec_r^{\mathcal{O}}$ is a well-founded strict partial order on $\Delta^{\mathcal{O}}$.*

Proof We show that $\prec_r^{\mathcal{O}}$ is (*i*) transitive, (*ii*) irreflexive and (*iii*) antisymmetric.

(*i*) Suppose $(x, y) \in \prec_r^{\mathcal{O}}$ and $(y, z) \in \prec_r^{\mathcal{O}}$. Then $\exists (x, u) \in r^{\mathcal{O}}$ and $\exists (y, v) \in r^{\mathcal{O}}$ such that $(\forall (z, v') \in r^{\mathcal{O}})[((x, u), (y, v)) \in \ll_r^{\mathcal{O}}$ and $((y, v), (z, v')) \in \ll_r^{\mathcal{O}}]$. Since $\ll_r^{\mathcal{O}}$ is transitive, $(x, z) \in \prec_r^{\mathcal{O}}$. Hence $\prec_r$ is transitive.

(*ii*) Suppose $(x, x) \in \prec_r^{\mathcal{O}}$, then $\exists (x, y) \in r^{\mathcal{O}}$ such that $((x, y), (x, y)) \in \ll_r^{\mathcal{O}}$, which contradicts the irreflexivity of $\ll_r^{\mathcal{O}}$. Hence $\prec_r^{\mathcal{O}}$ is irreflexive.

(*iii*) Suppose $(x, y) \in \prec_r^{\mathcal{O}}$ and $(y, x) \in \prec_r^{\mathcal{O}}$. Then $\exists (x, z) \in r^{\mathcal{O}}$ and $\exists (y, u) \in r^{\mathcal{O}}$ such that $((x, z), (y, u)) \in \ll_r^{\mathcal{O}}$ and $((y, u), (x, z)) \in \ll_r^{\mathcal{O}}$, which contradicts the asymmetry of $\ll$. Hence $\prec_r^{\mathcal{O}}$ is asymmetric (antisymmetric and irreflexive).

That $\prec_r^{\mathcal{O}}$ is well-founded follows from the well-foundedness of $\ll_r^{\mathcal{O}}$. □

Lemma 2 *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, and let $\prec$ be a well-founded strict partial order on $\Delta^{\mathcal{O}}$. Let $\mathcal{O}'$ be an extension of $\mathcal{O}$ with fresh role name $r' \in \mathsf{R}$ added, such that:*

$$\mathcal{O}' \Vdash \top \sqsubseteq \exists r'.\top;$$
$$\ll_{r'}^{\mathcal{O}'} := \{((x, z), (y, v)) \mid x \prec y \text{ and } (x, z), (y, v) \in r'^{\mathcal{O}'}\}.$$

Let $\prec_{r'}^{\mathcal{O}'}$ be as in Definition 3. Then $\prec\; = \;\prec_{r'}^{\mathcal{O}'}$.

Proof Suppose $(x, y) \in \prec$. Then x and y are both in the domain of $r^{\mathcal{O}'}$, and $((x, z), (y, v)) \in \ll_r^{\mathcal{O}'}$ for all $(x, z), (y, v) \in r^{\mathcal{O}'}$. Hence $(x, y) \in \prec_r^{\mathcal{O}'}$. Conversely, suppose that $(x, y) \in \prec_r^{\mathcal{O}'}$. Then $(\exists (x, z) \in r^{\mathcal{O}'})(\forall (y, v) \in r^{\mathcal{O}'})[((x, z), (y, v)) \in \ll_r^{\mathcal{O}'}]$. Since y is in the domain of $r^{\mathcal{O}'}$, we have $(x, y) \in \prec$. □

Lemma 3 *For every $r \in \mathsf{R}$, $\precsim_r$ is a preferential subsumption relation on concepts in that the following rules (a.k.a. the DL version of the well-known KLM-style postulates or properties) hold for every ordered interpretation $\mathcal{O}$, i.e., whenever $\mathcal{O}$ satisfies the antecedent, it satisfies the consequent as well:*

(Ref) $C \precsim_r C$ (LLE) $\dfrac{C \equiv D,\ C \precsim_r E}{D \precsim_r E}$ (And) $\dfrac{C \precsim_r D,\ C \precsim_r E}{C \precsim_r D \sqcap E}$

(Or) $\dfrac{C \precsim_r E,\ D \precsim_r E}{C \sqcup D \precsim_r E}$ (RW) $\dfrac{C \precsim_r D,\ D \sqsubseteq E}{C \precsim_r E}$ (CM) $\dfrac{C \precsim_r D,\ C \precsim_r E}{C \sqcap D \precsim_r E}$

Proof We show that $\precsim_r$ is preferential for every ordered interpretation $\mathcal{O}$.

(Ref): Let $x \in \Delta^{\mathcal{O}}$ be such that $x \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$. Then $x \in C^{\mathcal{O}}$ and therefore $\mathcal{O} \Vdash C \precsim_r C$.

(LLE): Assume that $\mathcal{O} \Vdash C \precsim_r E$ and $\mathcal{O} \Vdash C \equiv D$. Then $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq E^{\mathcal{O}}$ and $C^{\mathcal{O}} = D^{\mathcal{O}}$. It follows that $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} = \min_{\prec_r^{\mathcal{O}}} D^{\mathcal{O}}$. Hence $\min_{\prec_r^{\mathcal{O}}} D^{\mathcal{O}} \subseteq E^{\mathcal{O}}$, and therefore $\mathcal{O} \Vdash D \precsim_r E$.

(And): Assume that $\mathcal{O} \Vdash C \precsim_r D$ and $\mathcal{O} \Vdash C \precsim_r E$. Then $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq D^{\mathcal{O}}$ and $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq E^{\mathcal{O}}$, and then $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq D^{\mathcal{O}} \cap E^{\mathcal{O}}$, from which follows $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq (D \sqcap E)^{\mathcal{O}}$. Hence $\mathcal{O} \Vdash C \precsim_r D \sqcap E$.

(Or): Assume that $C \precsim_r E$ and $D \precsim_r E$. Let $x \in \min_{\prec_r^{\mathcal{O}}}(C \sqcup D)^{\mathcal{O}}$. Then x is minimal in $C^{\mathcal{O}} \cup D^{\mathcal{O}}$, and therefore either $x \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$ or $x \in \min_{\prec_r^{\mathcal{O}}} D^{\mathcal{O}}$. In either case $x \in E^{\mathcal{O}}$. Hence $\mathcal{O} \Vdash C \sqcup D \precsim_r E$.

(RW): Assume that $\mathcal{O} \Vdash C \precsim_r D$ and $\mathcal{O} \Vdash D \sqsubseteq E$. Then $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq D^{\mathcal{O}}$ and $D^{\mathcal{O}} \subseteq E^{\mathcal{O}}$. Hence $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq E^{\mathcal{O}}$ and then $\mathcal{O} \Vdash C \precsim_r E$.

(CM): Assume that $\mathcal{O} \Vdash C \precsim_r D$ and $\mathcal{O} \Vdash C \precsim_r E$. Then $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq D^{\mathcal{O}}$ and $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq E^{\mathcal{O}}$. Let $x \in \min_{\prec_r^{\mathcal{O}}}(C \sqcap D)^{\mathcal{O}}$. We show that $x \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$.

Suppose this is not the case. Since $\prec_r^{\mathcal{O}}$ is well-founded, there must be $x' \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$ such that $x' \prec_r^{\mathcal{O}} x$. Because $\mathcal{O} \Vdash C \precsim_r D$, $x' \in D^{\mathcal{O}}$, and then $x' \in C^{\mathcal{O}} \cap D^{\mathcal{O}}$, i.e., $x' \in (C \sqcap D)^{\mathcal{O}}$. From this and $x' \prec_r^{\mathcal{O}} x$ it follows that x is not minimal in $(C \sqcap D)^{\mathcal{O}}$, which is a contradiction. Hence $x \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$. From this and $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq E^{\mathcal{O}}$, it follows that $x \in E^{\mathcal{O}}$. Hence $\mathcal{O} \Vdash C \sqcap D \precsim_r E$. □

Lemma 5 *Let $\mathcal{O}$ be a modular interpretation, $r \in \mathsf{R}$ and let $\prec_r^{\mathcal{O}}$ be as in Definition 3. Then $\prec_r^{\mathcal{O}}$ is a modular order.*

Proof It follows from Lemma 1 that $\prec_r^{\mathcal{O}}$ is a well-founded strict partial order. To prove modularity, we show that the incomparability relation of $\prec_r^{\mathcal{O}}$ is transitive:

Let $\overline{x} := \min_{\ll_r^{\mathcal{O}}} r^{\mathcal{O}|x}$. That is, $\overline{x}$ is the set of $\ll_r^{\mathcal{O}}$-minimal elements of $r^{\mathcal{O}}$ restricted to domain $\{x\}$. Now suppose x and y are incomparable in $\prec_r^{\mathcal{O}}$, and y and z are incomparable in $\prec_r^{\mathcal{O}}$. Since $\ll_r^{\mathcal{O}}$ is modular, all elements of $\overline{x} \cup \overline{y}$ are incomparable in $\ll_r^{\mathcal{O}}$. Similarly, all elements of $\overline{y} \cup \overline{z}$ are incomparable. It then follows from the modularity of $\ll_r^{\mathcal{O}}$ that all elements of $\overline{x} \cup \overline{z}$ are incomparable in $\ll_r^{\mathcal{O}}$. Therefore x and z are incomparable in $\prec_r^{\mathcal{O}}$. □

Lemma 6 *For every $r \in \mathsf{R}$, $\precsim_r$ is a rational subsumption relation on concepts in that every modular interpretation $\mathcal{O}$ satisfies the following rational monotonicity property:*

$$(\text{RM})\ \frac{C \precsim_r E,\ C \not\precsim_r \neg D}{C \sqcap D \precsim_r E}$$

Proof Assume we have that $\mathcal{O} \Vdash C \precsim_r E$ and $\mathcal{O} \Vdash C \not\precsim_r \neg D$. From the latter it follows that there is $x \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$ such that $x \in D^{\mathcal{O}}$, and hence $x \in (C \sqcap D)^{\mathcal{O}}$. Let $x' \in \min_{\prec_r^{\mathcal{O}}}(C \sqcap D)^{\mathcal{O}}$. We shall first prove that $x' \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$. Since $x \in (C \sqcap D)^{\mathcal{O}}$, $x \not\prec_r^{\mathcal{O}} x'$. That is, either $x' \prec_r^{\mathcal{O}} x$ or x and x' are incomparable. Now suppose there is some $x'' \in C^{\mathcal{O}}$ such that $x'' \prec_r^{\mathcal{O}} x'$. Since $\prec_r^{\mathcal{O}}$ is modular, $x'' \prec_r^{\mathcal{O}} x$, which contradicts the minimality of x in $C^{\mathcal{O}}$. Therefore $x' \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$. From $x' \in \min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}}$ and $\mathcal{O} \Vdash C \precsim_r E$ follows $x' \in E^{\mathcal{O}}$. Hence $\mathcal{O} \Vdash C \sqcap D \precsim_r E$. □

Lemma 10 *Let Δ be a fixed, countably infinite set. For every $\mathcal{KB}$, every $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $r \in \mathsf{R}$, $\mathcal{KB} \models_{\mathsf{mod}} C \precsim_r D$ if and only if $\mathcal{O} \Vdash C \precsim_r D$, for every $\mathcal{O} \in \mathit{Mod}_{\Delta}(\mathcal{KB})$.*

Proof Let Δ be a countably infinite domain. For the only-if part, if $\mathcal{KB} \models_{\mathsf{mod}} C \precsim_r D$, then obviously $\mathcal{O} \Vdash C \precsim_r D$ for every $\mathcal{O} \in \mathit{Mod}_{\Delta}(\mathcal{KB})$. For the if part, assume $\mathcal{KB} \not\models_{\mathsf{mod}} C \precsim_r D$. Then, thanks to Lemma 8, there is a modular interpretation $\mathcal{O}_{\text{fin}}$ with a finite domain that is a model of $\mathcal{KB}$ and a counter-model of $C \precsim_r D$. Given $\mathcal{O}_{\text{fin}}$, we can extend it to a model of $\mathcal{KB}$ that is a counter-model of $C \precsim_r D$ with a countably infinite domain. Now, let $\mathcal{O}' = \langle \Delta', \cdot^{\mathcal{O}'}, \ll^{\mathcal{O}'} \rangle$ be a modular model of $\mathcal{KB}$ and a counter-model of $C \precsim_r D$, with Δ' countably infinite. It is easy to build an isomorphic modular interpretation $\mathcal{O} = \langle \Delta, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, once we have defined a bijection $b : \Delta' \longrightarrow \Delta$, which must exist, being both Δ' and Δ countably infinite sets. We can define $\cdot^{\mathcal{O}}$ and $\ll^{\mathcal{O}}$ in the following way:

- For every $A \in \mathsf{C}$ and every $x \in \Delta'$, $b(x) \in A^{\mathcal{O}}$ iff $x \in A^{\mathcal{O}'}$;
- For every $r \in \mathsf{R}$ and every $x, y \in \Delta'$, $(b(x), b(y)) \in r^{\mathcal{O}}$ iff $(x, y) \in r^{\mathcal{O}'}$;
- For every $r \in \mathsf{R}$ and every $x, y, v, z \in \Delta'$, $(b(x), b(y)) \ll_r^{\mathcal{O}} (b(v), b(z))$ iff $(x, y) \ll_r^{\mathcal{O}'} (v, z)$.

It is easy to prove by induction on the construction of the concepts that, for every $C \in \mathcal{L}_{d\mathcal{ALC}}$ and every $x \in \Delta'$, $x \in C^{\mathcal{O}'}$ iff $b(x) \in C^{\mathcal{O}}$. Moreover, note that $x \prec_r^{\mathcal{O}'} y$ iff $b(x) \prec_r^{\mathcal{O}} b(y)$, where $\prec_r^{\mathcal{O}'}$ and $\prec_r^{\mathcal{O}}$ are obtained respectively from $\ll_r^{\mathcal{O}'}$ and $\ll_r^{\mathcal{O}}$ as in Definition 3. Therefore, $x \in \min_{\prec_r^{\mathcal{O}'}}(C^{\mathcal{O}'})$ iff $b(x) \in \min_{\prec_r^{\mathcal{O}}}(C^{\mathcal{O}})$. Hence, there is a $\mathcal{KB}$-model which is a counter model for $C \sqsubseteq\!\!\!\!\sim_r D$ with Δ as its domain. □

Lemma 11 *If $\mathcal{KB}$ is modularly satisfiable, then $\mathcal{O}_{\oplus}^{\mathcal{KB}}$ is a modular interpretation.*

Proof Since $\mathcal{KB}$ is modularly satisfiable, $\mathit{Mod}_{\Delta}(\mathcal{KB}) \neq \emptyset$, and therefore $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \neq \emptyset$. Since Δ is countably infinite, and each model is defined over a finite vocabulary, there are only countably many such models. The disjoint union of domains is therefore also countably infinite. Moreover, it is easy to see that (i) for every $A \in \mathsf{C}$, $A^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \subseteq \Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$; (ii) for every $r \in \mathsf{R}$, $r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \subseteq \Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \times \Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$, and (iii) for every $a \in \mathsf{I}$, $a^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \in \Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$. Finally, from the definitions of $\mathit{Mod}_{\Delta}(\mathcal{KB})$ and of $h_{\mathcal{O}}(x, y, r)$ and the construction of $\ll_r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$, it follows that $\ll_r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ is also a modular well-founded order. □

Lemma 14 *Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base, let $C \in \mathcal{L}_{d\mathcal{ALC}}$, let $r \in \mathsf{R}$ be satisfiable, and let $i \leq \omega$. If $\mathit{rank}_{\mathcal{KB}}(C, r) = i$, then $h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(C, r) = i$.*

Proof First, assume $i \neq \omega$. Given $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$, let $\mathit{rank}_{\mathcal{KB}}(C, r) = i$, and let $\mathcal{D}_{\geq i}^{rank}$ be the subset of $\mathcal{D}$ containing the defeasible subsumption statements with a ranking value of at least i. Let $\mathcal{O}'$ be a modular model of $\mathcal{T} \cup \mathcal{D}_{\geq i}^{rank}$ such that $h_{\mathcal{O}'}(C, r) = 0$. Such a model must exist, since $\mathit{rank}_{\mathcal{KB}}(C, r) = i$, that is, C is not exceptional in $\mathcal{T} \cup \mathcal{D}_{\geq i}^{rank}$ (cf. Definition 12). We can assume that $\mathcal{O}'$ has a finite domain, given the finite-model property (Lemma 7). For each defeasible subsumption $C \sqsubseteq\!\!\!\!\sim_r D \in \mathcal{D} \setminus \mathcal{D}_{\geq i}^{rank}$, let $\mathcal{O}_{C \sqcap D} \in \mathit{Mod}_{\Delta}(\mathcal{KB})$ be a model of $\mathcal{KB}$ satisfiying $C \sqcap D$. (It can be shown by induction that such a model exists for each such defeasible subsumption.) We define a new interpretation $\mathcal{O}'' = \langle \Delta^{\mathcal{O}''}, \cdot^{\mathcal{O}''}, \ll^{\mathcal{O}''} \rangle$ in the following way:

1. $\Delta^{\mathcal{O}''}$ is the disjoint union of the domains $\Delta^{\mathcal{O}_{C \sqcap D}}$, for every $C \sqsubseteq\!\!\!\!\sim_r D \in \mathcal{D} \setminus \mathcal{D}_{\geq i}^{rank}$, together with $\Delta^{\mathcal{O}'}$;
2. For every $A \in \mathsf{C}$ and every $x \in \Delta^{\mathcal{O}''}$, $x \in A^{\mathcal{O}''}$ if one of the two following cases holds: either $x \in \Delta^{\mathcal{O}_{C \sqcap D}}$ for some $C \sqsubseteq\!\!\!\!\sim_r D \in \mathcal{D} \setminus \mathcal{D}_{\geq i}^{rank}$ and $x \in A^{\mathcal{O}_{C \sqcap D}}$, or $x \in \Delta^{\mathcal{O}'}$ and $x \in A^{\mathcal{O}'}$;
3. For every $r \in \mathsf{R}$ and every $x, y \in \Delta^{\mathcal{O}''}$, $(x, y) \in r^{\mathcal{O}''}$ if one of the two following cases holds: either $x, y \in \Delta^{\mathcal{O}_{C \sqcap D}}$ for some $C \sqsubseteq\!\!\!\!\sim_r D \in \mathcal{D} \setminus \mathcal{D}_{\geq i}^{rank}$ and $(x, y) \in r^{\mathcal{O}''}$, or $x, y \in \Delta^{\mathcal{O}'}$ and $(x, y) \in r^{\mathcal{O}'}$;
4. For every $r \in \mathsf{R}$ and every $(x, y) \in r^{\mathcal{O}''}$ (and we know $r^{\mathcal{O}''} \neq \emptyset$, by hypothesis), $h_{\mathcal{O}''}(x, y, r) = j$ if one of the two following cases holds:
   - either $x, y \in \Delta^{\mathcal{O}_{C \sqcap D}}$ for some $C \sqsubseteq\!\!\!\!\sim_r D \in \mathcal{D} \setminus \mathcal{D}_{\geq i}^{rank}$ and $h_{\mathcal{O}_{C \sqcap D}}(x, y, r) = j$,
   - or $x, y \in \Delta^{\mathcal{O}'}$ and $h_{\mathcal{O}'}(x, y, r) = j - i$.

The height of each object in $\Delta^{\mathcal{O}''}$ is defined as usual. In particular, notice that:

$(*)$ For every $x \in \Delta^{\mathcal{O}''}$, $h_{\mathcal{O}''}(x, r) = j$ if either $x \in \Delta^{\mathcal{O}_{C \sqcap D}}$ for some $C \sqsubseteq\!\!\!\!\sim_r D \in \mathcal{D} \setminus \mathcal{D}_{\geq i}^{rank}$ and $h_{\mathcal{O}_{C \sqcap D}}(x, r) = j$, or $x \in \Delta^{\mathcal{O}'}$ and $h_{\mathcal{O}'}(x, r) = j - i$.

It can be proven that $\mathcal{O}''$ is a model of $\mathcal{KB}$: first we prove by induction on the construction of concepts that, for every $D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $x \in \Delta^{\mathcal{O}''}$, $x \in D^{\mathcal{O}''}$ if and only if the corresponding object falls under D in the original model; then we prove that for every concept D and every $r \in \mathsf{R}$, $h_{\mathcal{O}''}(D, r) = \min\{h_{\mathcal{O}}(D, r) \mid \mathcal{O} \in \bigcup_{E \sqsubsetsim_r F \in \mathcal{D} \setminus \mathcal{D}^{rank}_{\geq i}} \{\mathcal{O}_{E \sqcap F}\} \cup \{\mathcal{O}'\}\}$ (which corresponds to the principle of presumption of typicality). Since we have assumed $h_{\mathcal{O}'}(C, r) = 0$, in the construction of $\mathcal{O}''$ (see $(*)$ above), we must have $j = i$, and therefore $h_{\mathcal{O}''}(C, r) = i$.

Since $\mathcal{D} \setminus \mathcal{D}^{rank}_{\geq j}$ is finite, $\mathcal{O}''$ is obtained from the composition of a finite set of models with Δ as domain and a model with a finite domain, that is, $\mathcal{O}''$ has a countably infinite domain. That implies that there is a model $\mathcal{O}_{\Delta}$ of $\mathcal{KB}$ that is isomorphic to $\mathcal{O}''$ and that has Δ as domain. So $\mathcal{O}_{\Delta}$ takes part in the construction of the big modular interpretation $\mathcal{O}^{\mathcal{KB}}_{\oplus}$, and therefore $h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C, r) = i$.

Assume now that $i = \omega$. Let j be any natural number, i.e., $j < \omega$. Moreover, let $\mathcal{O}'$ be a modular model of $\mathcal{T} \cup \mathcal{D}^{rank}_{\geq j}$, i.e., $\mathcal{O}'$ satisfies the statements in the original knowledge base minus those DCIs that have been removed up to rank $j - 1$ by the ranking procedure. Since C is exceptional in $\mathcal{T} \cup \mathcal{D}^{rank}_{\geq j}$, we have $\mathcal{T} \cup \mathcal{D}^{rank}_{\geq j} \models_{\mathsf{mod}} \top \sqsubsetsim_r \neg C$ (cf. Definition 12) and therefore the height of any object in $C^{\mathcal{O}'}$, if any, cannot be 0, and then we must have $h_{\mathcal{O}'}(C, r) > 0$. Hence C cannot have any typical object at height j, in any modular model of the original knowledge base. In particular, that is true of the big modular interpretation, and since j is a natural number, we must have $h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C, r) = \omega$. □

Theorem 1 *Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base. For every $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $r \in \mathsf{R}$, $C \sqsubsetsim_r D$ is in the rational closure of $\mathcal{KB}$ if and only if $\mathcal{KB} \models_{\mathsf{rat}} C \sqsubsetsim_r D$.*

Proof Let $\mathcal{KB}$ be a modularly satisfiable defeasible knowledge base and let $\mathcal{O}^{\mathcal{KB}}_{\oplus}$ be its associated big modular interpretation. Moreover, let $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and $r \in \mathsf{R}$.

First of all, we observe that, if C is disjoint from the domain of r, including if either r or C is unsatisfiable, then $\mathcal{O}^{\mathcal{KB}}_{\oplus} \Vdash C \sqsubsetsim_r D$ if and only if $\mathcal{O}^{\mathcal{KB}}_{\oplus} \Vdash C \sqsubseteq D$ if and only if $h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C \sqcap \neg D, r') = \omega$, where $h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C \sqcap \neg D, r')$ is the height of $C \sqcap \neg D$ in the context of a new role r' with domain $\Delta^{\mathcal{O}^{\mathcal{KB}}_{\oplus}}$ introduced as in Lemma 2.

For the only-if part, assume $C \sqsubsetsim_r D$ is in the rational closure of $\mathcal{KB}$. Then, by the definition, we have $rank_{\mathcal{KB}}(C \sqcap D, r) < rank_{\mathcal{KB}}(C \sqcap \neg D, r)$ or $rank_{\mathcal{KB}}(C \sqcap \neg D) = \omega$. From this and Lemma 14, it follows immediately that $h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C \sqcap D, r) < h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C \sqcap \neg D, r)$ or $h_{\mathcal{O}^{\mathcal{KB}}_{\oplus}}(C \sqcap \neg D, r') = \omega$, which by Corollary 4 gives us $\mathcal{O}^{\mathcal{KB}}_{\oplus} \Vdash C \sqsubsetsim_r D$, and therefore $\mathcal{KB} \models_{\mathsf{rat}} C \sqsubsetsim_r D$.

For the if part, we note that the converse of Lemma 14 follows directly from the lemma. The proof of the theorem can now be completed by following the argument above in the converse direction. □