Pertinence Construed Modally

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Keywords: Pertinence, modal logic, infra-modal entailment, non-explosiveness, non-monotonicity

1 Introduction

Classical logic is, in a sense, the logic of complete ignorance. Any arbitrary ('trivial') dilation of the set of (classical) valuations satisfying α yields a β such that $\alpha \models \beta$. One intuitive connotation of 'entailment' is that more, some additional relation of 'relevance' or 'pertinence', should hold between α and β.

If rather specific, such extra information is usually expressed either as syntactic rules or as semantic constraints, and typically involves an (often binary) relation on the set of sentences of the language. More generally and vaguely the 'extra' may be a desire to adapt classical entailment $\models$ in order to obtain an entailment relation which more closely resembles human reasoning as precipitated in natural language.

Here we follow a semantic rather than syntactic approach, and consider pertinence relations which can be seen as *infra-modal* in the following sense: Similar to infra-classical entailments which are obtained by trimming classical Boolean entailment, we obtain infra-modal entailments by trimming standard modal entailments.

One road to infra-classicality is well known, that of *substructural* logics [10], which weaken the generating engine of *axioms* and *inference rules* for producing entailment pairs (α, β). Here we follow, in a sense, the opposite strategy: we first demand that $\alpha \models \beta$, but then (invoking extra information in the meta-level) *more*, trimming down the set of entailment pairs to infra-modal consequence.

2 Logical Preliminaries

We work in a propositional modal language $\mathcal{L}$ over a set of propositional atoms $\mathfrak{P}$, together with the distinguished atom $\top$ (*verum*), and with the normal modal operator $\Box$ [3,6]. Given $\Box$ and its dual $\Diamond$, we can also speak of their *converse* operators, $\breve{\Box}$ and $\breve{\Diamond}$, respectively.

We are interested in the class of models having a *reflexive* accessibility relation. This defines the modal logic KT [6]. We employ the following version of *local consequence*:

Definition 2.1 Given a KT-model $\mathscr{M} = \langle W, R, V\rangle$ and formulas α and β, we say that α *entails* β *in* $\mathscr{M}$ (noted $\alpha \models^{\mathscr{M}} \beta$) if and only if for every $w \in W$, if $w \Vdash^{\mathscr{M}} \alpha$, then $w \Vdash^{\mathscr{M}} \beta$.

Given a class $\mathscr{C}$ of KT-models and formulas α, and β, if $\alpha \models^{\mathscr{M}} \beta$ for every $\mathscr{M} \in \mathscr{C}$, we say that α *entails* β *in* $\mathscr{C}$ (noted $\alpha \models^{\mathscr{C}} \beta$); if $\models^{\mathscr{M}} \alpha$ for every $\mathscr{M} \in \mathscr{C}$, we say that α is *valid* in $\mathscr{C}$ (noted $\models^{\mathscr{C}} \alpha$).

A specific class of models can be determined by imposing additional axiom schemas (e.g. transitivity, reflexivity, etc.) or by means of *global axioms* (formulas one wants to be valid in the class). Since the class of models we are working with will be made clear from the context, for the sake of readability we shall dispense with superscripts and just write $\alpha \models \beta$ instead of $\alpha \models^{\mathscr{C}} \beta$.

3 Modal Pertinent Entailment

In our new entailment of β by α, the condition that we impose upon the (previously wild) $\beta \wedge \neg\alpha$-worlds is that now each of them must be *accessible* from *some* α-world. This establishes the mutual pertinence of α and β to each other. But note that this is not to say that the pertinence is between *worlds*. It is rather between the *sets* of α- and β-worlds.

Definition 3.1 *α pertinently entails β in the* KT*-model $\mathscr{M}$* (noted $\alpha \mid\!<^{\mathscr{M}} \beta$) if and only if $\alpha \models^{\mathscr{M}} \beta$ and $\beta \models^{\mathscr{M}} \breve{\Diamond}\alpha$. *$\alpha$ pertinently entails β in the class $\mathscr{C}$* of KT-models (noted $\alpha \mid\!<^{\mathscr{C}} \beta$) if and only if for every $\mathscr{M} \in \mathscr{C}$, $\alpha \mid\!<^{\mathscr{M}} \beta$.

Proposition 3.2 *Given a class of* KT*-models $\mathscr{C}$, $\mid\!<^{\mathscr{C}} = \bigcap\{\mid\!<^{\mathscr{M}} \mid \mathscr{M} \in \mathscr{C}\}$.*

Intuitively, Definition 3.1 states that premiss α and consequence β are *mutually pertinent* if and only if α entails β and every β-world is accessible from *some* α-world — importantly, the $\beta \wedge \neg\alpha$-worlds. (The α-worlds are each accessible from itself.)

In the symbol $\mid\!<$, the '$<$' refers to the infra-modal aspect of the entailment, as opposed to the '$=$' in $\models$, since what we do, in a sense, with the extra condition in Definition 3.1, is to 'cull down' some of the pairs in $\models$, obtaining a subset thereof.

Given a premiss α, the set of consequences that α entails in our new relation are all the βs that lie between that particular α-premiss and $\breve{\Diamond}\alpha$, and hence form a sub-lattice (closed under conjunction and disjunction) of the Lindenbaum-Tarski algebra of the modal language.

Given a consequence β, the set of all those premisses α such that $\alpha \mid\!< \beta$ does not always constitute a sub-lattice of the Lindenbaum-Tarski algebra, since it is not, in general, closed under conjunction. But it is closed under disjunction: if $\alpha_1 \mid\!< \beta$ and $\alpha_2 \mid\!< \beta$, then $\alpha_1 \vee \alpha_2 \mid\!< \beta$.

The second part in Definition 3.1 adds 'pertinence' to the traditional modal entailment. It says: from every β-world we can look back to some world, possibly different from where we are, and from which we could have come, in which α is true. The pertinence resides in the fairly subtle relationship required between (i) the truth values of sentences, and (ii) the accessibility between worlds. Obviously, $\mid\!<$ is an infra-modal entailment relation: if $\alpha \mid\!< \beta$, then $\alpha \models \beta$.

Given a model $\mathscr{M} = \langle W, R, V\rangle$, $id_W \subseteq R \subseteq W \times W$. The *minimum* (with respect to $\subseteq$) case, i.e., in any subclass $\mathscr{C}$ of KT-models $\mathscr{M} = \langle W, R, V\rangle$ such that $R = id_W$, corresponds to the *maximum pertinence* of the relation $\mid\!<$, namely the case $\mid\!<= \equiv$ (i.e., logical equivalence), since now $\beta \models \breve{\Diamond}\alpha$ says that $\beta \models \alpha$. On the other hand, let $\models_<$ denote $\models \setminus \{(\bot, \beta) \mid \beta \not\equiv \bot\}$. Then the *maximum* case, i.e., in any subclass $\mathscr{C}$ of KT-models $\mathscr{M} = \langle W, R, V\rangle$ such that $R = W \times W$, corresponds to the *minimum pertinence* of $\mid\!<$, namely when $\mid\!<= \models_<$ (since now $\beta \models \breve{\Diamond}\alpha$ says that $\beta \not\equiv \bot$ implies $\alpha \not\equiv \bot$). Therefore we have:

Theorem 3.3 $\equiv \subseteq \mid\!< \subseteq \models_<$.

Non-explosiveness $\mid\!<$ is *non-explosive* in the strong sense that *falsum* is not omnigenerating, in fact, only self-generating: if $\bot \mid\!< \beta$, then $\beta \equiv \bot$. No contingent or tautological sentence is $\mid\!<$-entailed by a contradiction. More generally:

Theorem 3.4 *Let* $\alpha \mid\!\overset{\mathscr{C}}{<} \beta$. *Then if* $\overset{\mathscr{C}}{\models} \alpha \to \bot$, *then* $\overset{\mathscr{C}}{\models} \beta \to \bot$.

In other words, no sentence satisfiable in a class $\mathscr{C}$ of models is $\mid\!\overset{\mathscr{C}}{<}$-entailed by a sentence *unsatisfiable* in that class.

Tautologies The set of *pertinent tautologies* of the modal language is identical to the set of all modal tautologies:

Theorem 3.5 $\top \mid\!< \alpha$ *if and only if* $\top \models \alpha$.

Contraposition Classically and modally we have *contraposition*. Not so for $\mid\!<$, and proof by contradiction does not hold in general.

Deduction Theorem The classical meta-theorem called *deduction*, or by some authors the *Ramsey test* for conditionals ($\alpha \models \beta$ is equivalent to $\top \models \alpha \to \beta$), does not hold for $\mid\!<$. We have that $\alpha \mid\!< \beta$ implies $\top \mid\!< \alpha \to \beta$, but not conversely — unless every β-world is accessible from some α-world, which is precisely the pertinence aspect of the definition of $\mid\!<$.

We noted in Theorem 3.5 above that the sets of modal and of pertinent tautologies are identical. While modal entailment $\alpha \models \beta$ is equivalent to $\alpha \to \beta$ being valid, this is false for pertinent entailment.

In our approach it is not difficult to define a modal conditional connective which *does* satisfy the Ramsey test. We define the modal binary connective $\diamond\!\!\to$, called the *pertinent conditional*, as follows:

Definition 3.6 $\alpha \diamond\!\!\to \beta \equiv_{\text{def}} (\alpha \to \beta) \wedge (\beta \to \breve{\Diamond}\alpha)$.

Theorem 3.7 *$\alpha \mid\!< \beta$ if and only if $\mid\!< \alpha \diamond\!\!\rightarrow \beta$.*

Positive Paradox One of the specific *bêtes noires* of relevance and relevant logicians is what they call *positive paradox* and write as $\alpha \rightarrow (\beta \rightarrow \alpha)$. With the introduction of our (stricter) conditional $\diamond\!\!\rightarrow$, one question that naturally arises is whether we have a pertinent version of positive paradox. The answer, as expected, is 'no', as shown by the following result:

Proposition 3.8 *$\not\mid\!< \alpha \diamond\!\!\rightarrow (\beta \diamond\!\!\rightarrow \alpha)$.*

Corollary 3.9 *$\alpha \not\mid\!< \beta \diamond\!\!\rightarrow \alpha$.*

With regards to a sound and complete proof-theoretical counterpart for $\mid\!<$, we can resort to existing decision procedures, notably tableaux and resolution, for both conditions in Definition 3.1. Note also that pertinent entailment satisfies the rule *modus ponens* in the following sense:

Modus Ponens

$$\frac{\alpha \mid\!< \beta,\ \alpha \mid\!< \beta \rightarrow \gamma}{\alpha \mid\!< \gamma}$$

Moreover, it turns out that our modal pertinent entailment is *non-monotonic*:

Non-Monotonicity For the entailment $\mid\!<$, the following monotonicity rule *fails*:

$$\frac{\alpha \mid\!< \beta,\ \gamma \models \alpha}{\gamma \mid\!< \beta}$$

This result stands in contrast to one of the fundamental, albeit tacit, assumptions in the non-monotonic reasoning literature [4,8,9], viz. that non-monotonic entailment relations are *a priori* supra-classical, or, at least, are obtained by relaxing the underlying (possibly non-classical) monotonic entailment [1].

Substitution of Equivalents Let $\alpha \mid\!< \beta$ and γ be a subformula of α. Then, for every γ' such that $\models \gamma \leftrightarrow \gamma'$, we have that $\alpha' \mid\!< \beta$, where α' is obtained by uniformly substituting γ' for γ in α. Consequently, we do not have the *variable sharing property* required in relevance logics [7].

By also requiring the underlying class of models to be *transitive*, i.e., if instead of KT we work in the modal logic $\mathsf{S4}$ [6], then we get a pertinent entailment that satisfies some additional, and contextually desirable rules, notably:

Transitivity (Pertinent Left Strengthening)

$$\frac{\alpha \mid\!< \beta,\ \beta \mid\!< \gamma}{\alpha \mid\!< \gamma}$$

Besides the rules discussed here, pertinent entailment also satisfies a number of interesting properties studied in the context of non-monotonic reasoning, such as Cautious Monotonicity and Cut [5].

4 Pertinence in the Context of Causation

Consider the following variant of the Yale shooting problem, called the Walking Turkey Scenario [2]: Assume that we want to hunt a turkey, which may be alive or not, and which may either be walking around or not. In such a scenario we have one action, namely that of shooting the turkey with a gun. Our language has the propositions $\mathfrak{P} = \{s, a, w\}$. Let s be interpreted as "the turkey is shot"; a as "the turkey is alive"; and w as "the turkey is walking".

This time, our set of background assumptions is $\mathcal{B} = \{w \to a, s \to \neg a, \Diamond s\}$. The intuition behind $\mathcal{B}$ is that a walking turkey is alive; a shot turkey is dead; and it is possible to shoot the turkey.

Now suppose that we want to define a class of transitive models in which the background assumptions in $\mathcal{B}$ are valid. First we make sure that the axiom schema 4 ($\Box\alpha \to \Box\Box\alpha$) [6] holds, and then we cull down the transitive models in which the formulas in $\mathcal{B}$ are *not* valid.

Here we are interested in entailments of the form: given that β is observed, is α the *cause* of β? While classically we have $\neg a \wedge \neg w \models \neg a$ and $\neg a \wedge \neg w \models \neg w$, we now get $\neg a \wedge \neg w \mid\!< \neg a$, but $\neg a \wedge \neg w \not\mid\!< \neg w$: the turkey could be alive and still prefer not to walk! We also do not have $a \wedge \Box\neg s \mid\!< a$ (explanation incompatible with background assumption — cf. Theorem 3.4). On the other hand, we do have $a \wedge \Box\Diamond s \mid\!< a$ (substitution of equivalents, since $\Box\Diamond s$ is valid in this class of models). Moreover, we have neither $s \mid\!< \neg a$ nor $\neg a \mid\!< s$: being shot is not the only possible cause for the death of the turkey; and being already dead does not explain anything at all for being shot.

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