Defeasible Linear Temporal Logic

Anasse Chafik[a], Fahima Cheikh-Alili[a], Jean-François Condotta[a] and Ivan Varzinczak[b]

[a]CRIL, Université d'Artois et CNRS, Lens, France; [b]LIASD, Université Paris 8, Paris, France

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ABSTRACT
After the seminal work of Kraus, Lehmann and Magidor (formally known as the KLM approach) on conditionals and preferential models, many aspects of defeasibility in more complex formalisms have been studied in recent years. Examples of these aspects are the notion of typicality in description logic and defeasible necessity in modal logic. We discuss a new aspect of defeasibility that can be expressed in the case of temporal logic, which is the normality in an execution. In this contribution, we take Linear Temporal Logic (LTL) as case study for this defeasible aspect. LTL has found extensive applications in Computer Science and Artificial Intelligence, notably as a formal framework for representing and verifying computer systems that vary over time. However, some systems may presents exceptions at some innocuous time points where they can be tolerated, or conversely, exceptions at other crucial time points where they need to be addressed. In order to ensure the reliability of such systems, we study a preferential extension of LTL, called defeasible linear temporal logic ($LTL^{\sim}$). In the first part of this paper, we show how semantics of KLM's preferential models can be integrated with LTL. We also discuss the addition of non-monotonic temporal operators as a way to formalise defeasible properties of these systems. The second part of this paper is a study of the satisfiability problem of $LTL^{\sim}$ sentences. Based on Sistla and Clarke's work on the complexity of the classical LTL language, we show the bounded-model property of two fragments of $LTL^{\sim}$ language. Moreover, we provide a procedure to check the satisfiability of sentences in both of these fragments.

KEYWORDS
Knowledge Representation, non-monotonic reasoning, temporal logic.

1. Introduction

Linear Temporal Logic (LTL) was introduced by Pnueli (15) as a formal tool for reasoning about executions of programs. It is a formalism in the family of temporal modal logics that uses modalities such as $\Box$ (always) and $\Diamond$ (eventually) to describe a program's execution history. The logic LTL is used for systems verification (16). With advances in technologies, many extensions have been developed during the years to better express behavioral changes of systems (8; 13; 21). One of such behavior is managing and tolerating exceptions within a system. In fact, computer systems are not either 100% secure or 100% defective, and the properties we wish to check may have innocuous and tolerable exceptions, or conversely, exceptions that must be carefully

CONTACT Anasse Chafik. Email: chafik@cril.fr

addressed in order to guarantee the overall reliability of the system. Similarly, the expected behavior of a system may be correct not for all possible executions, but rather for its most 'normal' or expected executions.

For the sake of argument, we consider a system that has exceptional states and there is a run that goes through these states. We can express the run using a linear sequence of time points, each of which is a state of the system. We can then describe the properties of the run using the LTL language. For example, the safety check (the error e will never occur) can be expressed with the sentence $\Box\neg e$. The particularity of this execution is while the error e is not present in the normal states, some of the aforementioned exceptional states have the error e. In this case, the safety check fails due to the presence of the error in these states. However, knowing that these states are exceptional, the presence of e in them is not what matters. We want to make sure that the safety check succeeds in the normal points of time instead of all of them. Hence, we need a "defeasible" version of the operator $\Box$ that ignores the check at some points of time that are deemed to be benign in a run.

It turns out that LTL, by virtue of being a formalism of the so-called classical type, whose underlying reasoning is that of mathematics and not that of common sense, does not allow at all to formalise the different nuances of the exceptions and even less to treat them. First of all, at the level of the object language (that of the logical symbols), it has operators behaving monotonically, and at the level of reasoning, possesses a notion of logical consequence which is monotonic too, and consequently, it is not adapted to the evolution of defeasible facts.

On the other hand, defeasible reasoning is studied in the field of non-monotonic reasoning (NMR). It has been widely investigated by philosophers and the AI community (12; 14; 20) for over 40 years now. One aspect of defeasibility is formalising and reasoning with the presence of exceptions. Such is the case of the conditional logics of Kraus, Lehmann, and Magidor (11) known as the KLM approach. In their approach, defeasible consequence relations $\mid\!\sim$ with a preferential semantics (17; 18) are studied. A conditional is a statement of the form $\alpha \mid\!\sim \beta$ and indicates that "normally, if α is true, then β is true". This assertion focuses solely on the normal worlds of α to satisfy β, leaving the exceptional worlds of α to not satisfy β. The aspect of defeasibility that copes with exceptions is extended to many classical logics from the propositional logics (2; 10; 11) to more complex frameworks (3; 4; 9).

In the case of modal logic, Britz and Varzinczak (4) use the KLM approach to describe a new defeasible aspect of non-monotonic reasoning, which is the "normal outcome of an action". In classical modal logic, the system goes from a state s to another state s' as result of the action a. In addition, we can describe the outcome of actions using modalities such as $\Box, \Diamond$. For example, the sentence $\Box\alpha$ is true in a state s if all the reachable states s' through the action a are states where α is true. However, in the presence of exceptional states in the system, one might say that the normal outcome of an action a is α. Specifically, from a starting state s, all normal reachable states s' as a result of an action a satisfy α, and leaves it open for the exceptional states s'' of the action a to not satisfy it. We can shift the notion of defeasibility from a premise of inference $\mid\!\sim$ to effect of an action. In order to do so, Britz and Varzinczak (4) defined defeasible versions of the necessity $\Box$ and possibility $\Diamond$ operators in order to express the normality of the action's outcome. In LTL, this aspect of defeasibility manifests itself as "normality during a run". If a run goes through exceptional states at some points of time, it is not required to uphold defeasible properties.

That is why we introduced an extended formalism of LTL, called defeasible linear temporal logic ($LTL^{\sim}$) (5; 6). It uses the preferential approach of KLM to non-monotonic reasoning (11). The defeasible aspect of $LTL^{\sim}$ adds a new dimension to the verification of a program's execution. We can, for instance, order time points from the important ones, which we call *normal*, to the lesser and lesser ones. *Normality* in LTL indicates the importance of a time point within an execution compared to others. We also introduced defeasible versions of the modalities *always* and *eventually*. With these defeasible modalities, we can express properties similar to their classical counterparts, targeting the most normal time points within the execution.

In this paper, we define a logical framework for reasoning about defeasible properties of program executions. We discuss the integration of preferential semantics in the case of LTL. The remainder of the present paper is structured as follows: In Section 2, we recall the logic LTL and the KLM approach to NMR. In Sections 3, 4 and 5, we set up the notation for the syntax and appropriate semantics of defeasible linear temporal logic ($LTL^{\sim}$). In Section 6, we discuss the properties of defeasible temporal operators. In Section 7, we highlight two fragments of the language that we shall use for this paper. In Sections 8, 9, 10, 11, 12 and 13, we investigate the satisfiability problem of the aforementioned fragments. We conclude this paper in Section 14. In order to lighten the main text, proofs of propositions and lemmas that are not present in the main text can be found in the Appendix.

2. Preliminaries

2.1. Linear Temporal Logic

Linear Temporal Logic was introduced by Pnueli (15) as a formal tool for hardware and software specification and verification. This formalism allows for the description of a program's executions. LTL is a modal temporal logic, it uses modalities to refer to time. We can encode sentences that describe the future of an execution, e.g., from now on a statement is *always* true, or, will *eventually* hold.

In this section, we highlight the syntax of LTL that we use throughout the paper. Let $\mathcal{P}$ be a finite set of *propositional atoms*. The set of operators in LTL can be split into two parts: the set of *Boolean connectives* $\neg, \wedge, \vee$, and that of *temporal operators* $\Box, \Diamond, \bigcirc, \mathcal{U}$, where $\Box$ reads as *always*, $\Diamond$ as *eventually*, $\bigcirc$ as *next* and $\mathcal{U}$ as *until*. The set of well-formed sentences expressed in LTL is denoted by $\mathcal{L}$. Sentences of $\mathcal{L}$ are built according to the following grammar:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \alpha\,\mathcal{U}\,\alpha$$

The temporal structure is a chronological linear succession of time points. We use the set of natural numbers equipped with $<$ in order to label each of these time points, i.e., $(\mathbb{N}, <)$. A *temporal interpretation* I is defined by a mapping function $V : \mathbb{N} \longrightarrow 2^{\mathcal{P}}$ which associates each time point $t \in \mathbb{N}$ of the temporal structure with a set of propositional atoms $V(t)$ that are true in t. (Propositions not belonging to $V(t)$ are assumed to be false at the given time point.) The truth conditions of LTL sentences are defined as follows, where I is a temporal interpretation and t a time point in $\mathbb{N}$:

- $I, t \models p$ if $p \in V(t)$;
- $I, t \models \neg\alpha$ if $I, t \not\models \alpha$;
- $I, t \models \alpha \wedge \alpha'$ if $I, t \models \alpha$ and $I, t \models \alpha'$;
- $I, t \models \alpha \vee \alpha'$ if $I, t \models \alpha$ or $I, t \models \alpha'$;
- $I, t \models \Box\alpha$ if $I, t' \models \alpha$ for all $t' \in \mathbb{N}$ s.t. $t' \geq t$;
- $I, t \models \Diamond\alpha$ if $I, t' \models \alpha$ for some $t' \in \mathbb{N}$ s.t. $t' \geq t$;

- $I, t \models \bigcirc\alpha$ if $I, t+1 \models \alpha$;
- $I, t \models \alpha\,\mathcal{U}\,\alpha'$ if $I, t' \models \alpha'$ for some $t' \geq t$ and for all $t \leq t'' < t'$ we have $I, t'' \models \alpha$.

Temporal interpretations are used to model the linear changes of a program over time. We use the term run to depict a possible sequence of an execution of a program (these sequences are represented by a temporal interpretation).

In a run, we are able to capture properties by expressing sentences about the current and future time points. The LTL language is used to express these properties. Here are some properties that can be expressed in LTL.

- Safety: $\Box\alpha$ means that the property α will always hold, from this point of the execution onwards.
- Liveness: $\Diamond\alpha$ means that the property α will hold eventually. In other words, at some future time point of the run, α is true.
- Response: $\Box\Diamond\alpha$ means that for any time point in the run there is a later time point where α holds.
- Persistence: $\Diamond\Box\alpha$ means that there exists a point in the run such that from then and onwards, α holds.

We say $\alpha \in \mathcal{L}$ is *satisfiable* if there are I and $t \in \mathbb{N}$ such that $I, t \models \alpha$. A sentence α is *valid* if for all temporal interpretations I and all $t \in \mathbb{N}$, we have $I, t \models \alpha$.

Sistla and Clarke (19) proved that the satisfiability in LTL to be a PSPACE-complete problem. Moreover, the satisfiability checking of many fragments of the language were investigated (7; 19). Table 1 contains the complexity of some notable fragments. The notation $L(O_1, O_2, \ldots, O_k)$ denotes that the language of LTL is restricted to the temporal operators between parenthesis. L_{NNF} indicates that the negation is allowed on the atomic propositions only.

Fragment	Satisfiability
$L(\Diamond)$	NP-complete
$L_{NNF}(\Diamond, \bigcirc)$	NP-complete
$L(\Diamond, \bigcirc)$	PSPACE-complete
$L(\mathcal{U})$	PSPACE-complete
$L(\mathcal{U}, \bigcirc)$	PSPACE-complete

Table 1.: Complexity of some LTL fragments.

2.2. KLM approach to non-monotonic reasoning

Non-monotonic reasoning covers a family of formalisms and logics that capture and represent *defeasible inference*. Using defeasible inference, reasoners draw conclusions even when the information is incomplete and they reserve the right to retract said conclusion in the light of further information. This deductive type of reasoning is similar to the common sense reasoning, it is used in philosophical fields and expert fields (e.g. suspects list during an investigation, medical diagnoses ...). However, classical logic (ranging from propositional to more complex formalisms like modal and description logic) fails to capture this aspect of *defeasibility* in inferences. Classical (or monotonic) inferences are by nature based on complete information and thus do not allow for the retraction of inferences.

Many aspects of defeasible reasoning have been studied in the literature of non-monotonic logic. Non-monotonic inferences of the form $\alpha \mid\!\sim \beta$ have the following meaning: "the normal worlds of α are worlds of β". The aforementioned statement expresses that from the most plausible, desired or in general normal worlds of α, we can infer β; and leaves open the α-worlds that are exceptional to not satisfy β.

The approach we highlight in this paper is the approach of Kraus, Lehmann et Magidor (11) (known as the KLM approach) to non-monotonic reasoning (the preferential system **P**). A propositional *defeasible consequence relation* $\mid\!\sim$ (11) is defined as a binary relation on sentences of an underlying propositional logic. The semantics of preferential consequence relation is in terms of *preferential models*: A preferential model on a set of atomic propositions $\mathcal{P}$ is a tuple $\mathscr{P} \stackrel{\text{def}}{=} (S, l, \prec)$ where S is a set of elements called worlds, $l : S \longrightarrow 2^{\mathcal{P}}$ is a mapping which assigns to each state s a single world $m \in 2^{\mathcal{P}}$ and $\prec$ is a *strict partial* order on S satisfying smoothness condition. In this setting, the states that are lower down in the ordering $\prec$ are more plausible, normal or in a general case preferred, than those that are higher up.

Let α be a propositional sentence, the notation $[\![\alpha]\!]$ denotes the set of worlds $s \in S$ that satisfy α, called α-worlds. The set $\min_{\prec}([\![\alpha]\!])$ are α-worlds that are minimal with respect to the ordering relation $\prec$. The smoothness condition states that for any sentence α, the set $\min_{\prec}([\![\alpha]\!])$ is not empty (see Kraus et al. (11) for reference). This condition ensures that the set of minimal α-worlds has at least one minimal world. A statement $\alpha \mid\!\sim \beta$ is true if $\min_{\prec}([\![\alpha]\!]) \subseteq [\![\beta]\!]$. The conditional entailment $\alpha \mid\!\sim \beta$ holds in a preferential model iff the minimal α-worlds are also β-worlds.

Britz and Varzinczak (4) defined defeasible versions of necessity ($\Box$) and possibility ($\Diamond$) in modal logics and their role to describe the *normality* of an action. For example, suppose that we want to toggle a light, the light will turn on generally. Exceptionally, the light will not turn on. This can be either because the light bulb is broken or an overcharge resulted from switching the light. In the latter occurrence, the light bulb not lighting up is an exceptional outcome of the *action* of switching the light. *Normality* then may shift from the premise of the inference to the effect of an action. This gives the reasoner the power to express the normality at the language level and use it in the scope of other logical operators. We present briefly in this paper the preferential models in this case of modal logic and a new type of modalities, called *defeasible modalities.*

A *preferential Kripke model* is a tuple $\mathscr{P} \stackrel{\text{def}}{=} (S, R, V, \prec)$ where S is a set of states, R is the accessibility relation, $V : S \longrightarrow 2^{\mathcal{P}}$ is a valuation function and $\prec$ is a strict partial order on S that satisfies the smoothness condition. The language of defeasible modal logic is recursively defined as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \mathord{\approx\!\!\!\!\Box}\alpha \mid \mathord{\approx\!\!\!\!\Diamond}\alpha$$

The defeasible modality $\mathord{\approx\!\!\!\!\Box}$ reads as *defeasible necessity*, and $\mathord{\approx\!\!\!\!\Diamond}$ reads *defeasible possibility.* We shall discuss the truth values behind these defeasible operators next.

Let $S' \subseteq S$, then $min_{\prec}(S')$ denotes the set of minimal elements of S' with respect to $\prec$. Let $s \in S$, the set $R(s)$ denotes the elements of S that are accessible to s by the relation R. The truth values of modalities in defeasible modal logic are defined as follows.

Definition 2.1. Let $\mathscr{P} \stackrel{\text{def}}{=} (S, R, V, \prec)$ be a preferential Kripke model and $s \in S$.

- $\mathscr{P}, s \models \Box\alpha$ if $\mathscr{P}, s' \models \alpha$ for all s' in $R(s)$;
- $\mathscr{P}, s \models \Diamond\alpha$ if $\mathscr{P}, s' \models \alpha$ for some s' in $R(s)$.
- $\mathscr{P}, s \models \widetilde{\Box}\alpha$ if $\mathscr{P}, s' \models \alpha$ for all s' in $min_{\prec}(R(s))$;
- $\mathscr{P}, s \models \widetilde{\Diamond}\alpha$ if $\mathscr{P}, s' \models \alpha$ for some s' in $min_{\prec}(R(s))$.

The sentence $\widetilde{\Box}\alpha$ is true if all the minimal states that are accessible to s via R satisfy the sentence α. The sentence $\widetilde{\Diamond}\alpha$ is true if some minimal states that are accessible to s via R satisfy the sentence α. We can see that defeasible modalities behave in a similar fashion as their classical counterparts. In addition, defeasible modalities single out the preferred reachable states, by taking into account their order with respect to the relation $\prec$.

Since LTL also uses modalities such $\Box, \Diamond$ to refer to time, we investigate in this work, the integration of defeasible modalities to the LTL language. We present, in the upcoming section, their role in the context of the LTL formalism.

3. Defeasible LTL

In this section, we describe a formalism for reasoning about time that is able to handle exceptional points of time (5). We do so by investigating a defeasible extension of LTL with a preferential semantics. The following example introduces a case scenario we shall be using in the remainder of this section, with the purpose of giving a motivation for this formalism and better illustrating the definitions in what follows.

Example 3.1. We have a computer program in which the values of its variables change with time. In particular, the agent wants to check two parameters, say x and y. These two variables take one and only one value between 1 and 3 on each iteration of the program. We represent the set of atomic propositions by $\mathcal{P} = \{x_1, x_2, x_3, y_1, y_2, y_3\}$ where x_i (resp. y_i) for all $i \in \{1, 2, 3\}$ is true iff the variable x (resp. y) has the value i in a current iteration. Figure 1 depicts a temporal interpretation corresponding to a possible behaviour of such a program:

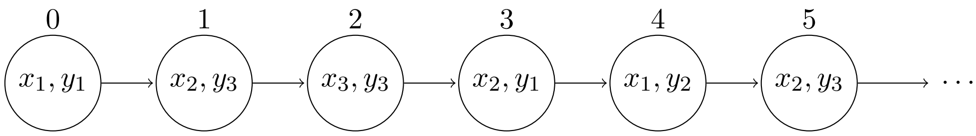


Figure 1.: LTL interpretation (for $t > 5$, $V(t) = V(5) = \{x_2, y_3\}$)

Under normal circumstances, the program assigns the value 3 to y whenever $x = 2$. We can express this fact using classical LTL as follows: $\Box(x_2 \to y_3)$, with $x_2 \to y_3$ defined by $\neg x_2 \lor y_3$. Nevertheless, the agent notices that there is one exceptional time point (the time point 3) where the program assigns the value 1 to y when $x = 2$.

Some might consider that the current program is defective at some points of time. In LTL, the statement $\Box(x_2 \to y_3) \land \Diamond(x_2 \land y_1)$ will always be false, since y cannot have two different values in an iteration where $x = 2$. Nonetheless we want to propose a logical framework that is exception tolerant for reasoning about a system's behaviour. We would like express that $(x_2 \to y_3)$ is true in all normal time points while taking into account that there might be some exceptional time points where $(x_2 \to y_3)$ is not necessarily true.

4. Introducing defeasible temporal operators

Britz and Varzinczak (4) introduced new modal operators called defeasible modalities. Defeasible operators, unlike their classical counterparts, are able to single out normal worlds from those that are less normal or exceptional in the reasoner's mind. Using a similar approach, we extend the vocabulary of classical LTL with the *defeasible temporal operators* ${\Box\!\!\!\!\sim}$ and ${\Diamond\!\!\!\!\sim}$. Sentences of the resulting logic $LTL^{\sim}$ are built up according to the following grammar:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \bigcirc\alpha \mid \alpha\,\mathcal{U}\,\alpha \mid {\Box\!\!\!\!\sim}\alpha \mid {\Diamond\!\!\!\!\sim}\alpha$$

Other standard Boolean operators are part of the syntax of $LTL^{\sim}$. Let α, β be two sentences of $LTL^{\sim}$ language, the symbol $\top$ is an abbreviation of $\alpha \vee \neg\alpha$, $\bot$ is an abbreviation of $\alpha \wedge \neg\alpha$, the implication operator is $\alpha \to \beta \overset{\text{def}}{=} \neg\alpha \vee \beta$ and the equivalence operator is $\alpha \leftrightarrow \beta \overset{\text{def}}{=} (\alpha \to \beta) \wedge (\beta \to \alpha)$. The intuition behind the defeasible operators in $LTL^{\sim}$ is the following: ${\Box\!\!\!\!\sim}$ reads as *defeasible always* and ${\Diamond\!\!\!\!\sim}$ reads as *defeasible eventuality.* The set of all well-formed $LTL^{\sim}$ sentences is denoted by $\mathcal{L}^{\sim}$. It is worth to mention that any well-formed sentence $\alpha \in \mathcal{L}$ is a sentence of $\mathcal{L}^{\sim}$. Here are some examples of well-formed sentences in $LTL^{\sim}$.

Example 4.1. Let $p, q \in \mathcal{P}$:

$$p, \quad \neg p, \quad \Box(p \wedge q) \to \Diamond p, \quad {\Diamond\!\!\!\!\sim}{\Box\!\!\!\!\sim}p, \quad \Box{\Diamond\!\!\!\!\sim}p \wedge {\Box\!\!\!\!\sim}\neg q$$

Same as the negation and temporal operators, defeasible operators have higher precedence than the other operators. As such, the sentence $\Box{\Diamond\!\!\!\!\sim}p \wedge {\Box\!\!\!\!\sim}\neg q$ is the same as $(\Box{\Diamond\!\!\!\!\sim}p) \wedge ({\Box\!\!\!\!\sim}\neg q)$ and not $\Box{\Diamond\!\!\!\!\sim}(p \wedge {\Box\!\!\!\!\sim}\neg q)$.

Example 4.2. Going back to Example 3.1, the sentence $\Box(x_2 \to y_3) \wedge \Diamond(x_2 \wedge y_1)$ cannot be true. Suppose that $x_2 \to y_3$ is *always* true. Therefore, $\Box(x_2 \to y_3)$ would then true and $\Diamond(x_2 \wedge y_1)$ would be false. Otherwise, if there is a time point such that $x_2 \wedge y_1$, then $\Diamond(x_2 \wedge y_1)$ would be true and $\Box(x_2 \to y_3)$ would be false (such is the case in Figure 1). Both of the sentences $\Box(x_2 \to y_3)$ and $\Diamond(x_2 \wedge y_1)$ cannot be true at the same time.

On the other hand, it is possible to express this specification using *defeasible always.* The sentence ${\Box\!\!\!\!\sim}(x_2 \to y_3)$ is true if $x_2 \to y_3$ is true all future *normal* time points. There might be some time points in the future where x_2 and y_1 are true and $\Diamond(x_2 \wedge y_1)$ is true. As long such time points are *exceptional*, the sentence ${\Box\!\!\!\!\sim}(x_2 \to y_3)$ remains true. As such, the sentence ${\Box\!\!\!\!\sim}(x_2 \to y_3) \wedge \Diamond(x_2 \wedge y_1)$ can still be true.

Normality can be expressed using non-monotonic operators. A similar version of the classical properties (see Section 2.1) can be expressed over runs that contain exceptional time points. These defeasible properties target future time points that are *normal* on one hand, and ignore states that are *exceptional* on the other. Here are some defeasible properties that can be expressed in $LTL^{\sim}$.

- Defeasible safety: ${\Box\!\!\!\!\sim}\alpha$ means that the property α holds for all normal future time points of the run.
- Pertinent liveness: ${\Diamond\!\!\!\!\sim}\alpha$ means that the property α will hold in a normal future time point of the run.
- Defeasible response: ${\Box\!\!\!\!\sim}{\Diamond\!\!\!\!\sim}\alpha$ means that for all normal time points of the run,

there is a later normal time point where α holds.

- Defeasible persistence: $\lozenge\!\!\!\sim \boxminus \alpha$ means that there exists a normal time point in the run such that α holds for all normal future time points.

The reasoner can therefore express defeasible properties using these new modalities, and more importantly, use it alongside the rest of other operators of $LTL^{\sim}$. Next we shall discuss how to interpret statements of $LTL^{\sim}$ formalism and how to determine the truth values of each well-formed sentence in $\mathcal{L}^{\sim}$.

5. Preferential interpretations

Moving on to the semantics, an $LTL^{\sim}$ interpretation I is a pair $I \stackrel{\text{def}}{=} (V, \prec)$. Recall that in Section 2.1, a temporal structure is represented by the ordering of integers $(\mathbb{N}, <)$. This shall not change for defeasible LTL interpretations. The function V is a valuation function which associates each time point $t \in \mathbb{N}$ with a truth assignment of all propositional atoms $V(t)$. The preferential component $\prec$ of the interpretation of $LTL^{\sim}$ is directly inspired by the preferential semantics proposed by Shoham (17) and used in the KLM approach (11). The preference relation $\prec$ is a strict partial order on points of time. Following the KLM approach (11), $t \prec t'$ means that t is more normal or preferred than t'. Time points can be ordered using the relation $\prec$, the closer they are to being minimal with respect to $\prec$, the more preferable they are, and vice versa, the farther they are to being minimal with respect to $\prec$, the more exceptional they become. We also use the notation $(t, t') \in \prec$ to indicate that t is more preferred than t'. The relation $\prec$ is an ordering relation for time points of a temporal structure, which symbolizes the preference over them. Before setting the formal definition for $LTL^{\sim}$ interpretation, we introduce the notion of minimality and well-foundness w.r.t. the relation $\prec$.

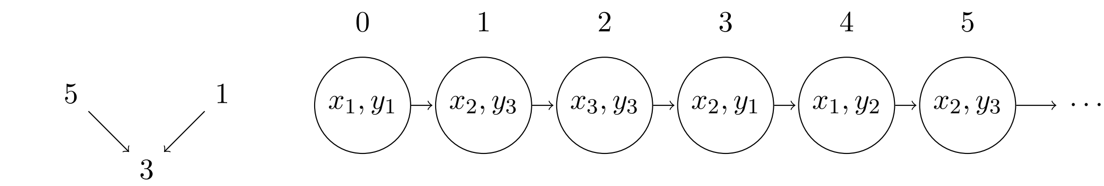
Definition 5.1. (Minimality w.r.t. $\prec$) Let $\prec$ be a strict partial order on $\mathbb{N}$ and $N \subseteq \mathbb{N}$. The set of the minimal elements of N w.r.t. $\prec$, denoted by $\min_{\prec}(N)$, is defined by $\min_{\prec}(N) \stackrel{\text{def}}{=} \{t \in N \mid \text{there is no } t' \in N \text{ such that } t' \prec t\}$.

Definition 5.2 (Well-founded set)**.** Let $\prec$ be a strict partial order on $\mathbb{N}$. We say $\mathbb{N}$ is *well-founded w.r.t.* $\prec$ iff $\min_{\prec}(N) \neq \emptyset$ for every $\emptyset \neq N \subseteq \mathbb{N}$.

Definition 5.3 (Preferential temporal interpretation)**.** An $LTL^{\sim}$ interpretation on a set of propositional atoms $\mathcal{P}$, also called preferential temporal interpretation on $\mathcal{P}$, is a pair $I \stackrel{\text{def}}{=} (V, \prec)$ where V is a mapping function $V : \mathbb{N} \longrightarrow 2^{\mathcal{P}}$, and $\prec \ \subseteq \mathbb{N} \times \mathbb{N}$ is a strict partial order on $\mathbb{N}$ such that $\mathbb{N}$ is well-founded w.r.t. $\prec$. We denote the set of preferential temporal interpretations by $\mathfrak{I}$.

Example 5.4. Going back to the run in Example 3.1, time points where $x = 2$ and $y = 3$ are more preferred than time points where $x = 2$ and $y = 1$. For example, the time point 1 and 5 are more preferred than 3. We extend the interpretation I in Example 3.1 by adding the set $\prec := \{(5, 3), (1, 3)\}$. Figure 2 represents a preferential temporal interpretation $I = (V, \prec)$ of the second run. Directed edges ($1 \longrightarrow 3$ for example) in Sub-figure a represent the pairs in the preference relation $\prec$.

In what follows, given an ordering relation $\prec$ and a time point $t \in \mathbb{N}$, the set of *preferred future time points relative to* t is the set $\min_{\prec}([t, \infty[)$ which is denoted in short by $\min_{\prec}(t)$. It is also worth pointing out that given a preferential interpretation



(a) Preference relation $\prec$ (b) For $t > 5$, $V(t) = V(5) = \{x_2, y_3\}$

Figure 2.: Preferential temporal interpretation $I = (V, \prec)$

$I = (V, \prec)$ and $\mathbb{N}$, the set $\min_{\prec}(t)$ is always a non-empty subset of $[t, \infty[$ at any time point $t \in \mathbb{N}$.

Preferential temporal interpretations provide us with an intuitive way of interpreting sentences of $\mathcal{L}^{\sim}$. Let $\alpha \in \mathcal{L}^{\sim}$, let $I = (V, \prec)$ be a preferential interpretation, and let t be a time point in $\mathbb{N}$. Satisfaction of α at t in I, denoted $I, t \models \alpha$, is defined as follows:

- $I, t \models \Box\!\!\!\!\sim\alpha$ if $I, t' \models \alpha$ for *all* $t' \in \min_{\prec}(t)$;
- $I, t \models \Diamond\!\!\!\!\sim\alpha$ if $I, t' \models \alpha$ for *some* $t' \in \min_{\prec}(t)$.

The truth values of Boolean connectives and classical modalities are defined as in LTL. The intuition behind a sentence of the form $\Box\!\!\!\!\sim\alpha$ is that α holds in *all* preferred time points that come after t. $\Diamond\!\!\!\!\sim\alpha$ intuitively means that α holds on at least one preferred time point relative to the future of t.

We say $\alpha \in \mathcal{L}^{\sim}$ is *preferentially satisfiable* if there is a preferential temporal interpretation I and a time point t in $\mathbb{N}$ such that $I, t \models \alpha$. Without loss of generality, we can say that $\alpha \in \mathcal{L}^{\sim}$ is preferentially satisfiable if there is a preferential temporal interpretation I s.t. $I, 0 \models \alpha$. A sentence $\alpha \in \mathcal{L}^{\sim}$ is *valid* (denoted by $\models \alpha$) if for all preferential temporal interpretations I and time points t in $\mathbb{N}$, we have $I, t \models \alpha$. We shall highlight the study of the satisfiability of $LTL^{\sim}$ sentences in the upcoming sections of this paper.

Example 5.5. In the interpretation $I = (V, \prec)$ in Figure 2, the set of future preferred time points relative to 0 is $\min_{\prec}(0) = \{0, 1, 2, 4\} \cup [5, \infty[$. We have the following:

- The time point 3 has the valuation $V(3) = \{x_2, y_1\}$. Thus, we have $I, 0 \models \Diamond(x_2 \wedge y_1)$ because $I, 3 \models x_2 \wedge y_1$. Moreover, we have $I, 0 \not\models \Box(x_2 \rightarrow y_3)$ because $I, 3 \not\models x_2 \rightarrow y_3$. Therefore, we conclude that $I, 0 \not\models \Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$. See that since x and y can have one and only one value, then the $\Box(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$ is *always* false (at most, either $\Box(x_2 \rightarrow y_3)$ or $\Diamond(x_2 \wedge y_1)$ but never both).
- Using defeasible temporal operators, we have $I, t \models x_2 \rightarrow y_3$ for all $t \in \min_{\prec}(0)$. See that the exceptional time point 3, on which the statement $x_2 \rightarrow y_3$ is false, is not in $\min_{\prec}(0)$. Therefore, we can infer that $I, 0 \models \Box\!\!\!\!\sim(x_2 \rightarrow y_3) \wedge \Diamond(x_2 \wedge y_1)$.

We can see that the addition of $\prec$ relation preserves the truth values of all classical temporal sentences. Moreover, for every $\alpha \in \mathcal{L}$, we have that α is satisfiable in LTL if and only if α is preferentially satisfiable in $LTL^{\sim}$.

6. Properties of defeasible temporal modalities

In this section, we discuss properties of defeasible temporal modalities next in relation to their classical temporal operators (1).

Proposition 6.1 (Duality)**.** *Let α be a well-formed sentence in $\mathcal{L}^{\sim}$. We have:*

$$\models \widetilde{\Box}\alpha \leftrightarrow \neg \widetilde{\Diamond}\neg\alpha$$

Proof. We take an arbitrary $I = (V, \prec) \in \mathfrak{I}$, $\alpha \in \mathcal{L}^{\sim}$ and $t \in \mathbb{N}$. For the only-if part, we assume that $I, t \models \widetilde{\Box}\alpha$ and suppose that $I, t \not\models \neg \widetilde{\Diamond}\neg\alpha$. Since $I, t \models \widetilde{\Box}\alpha$, we have $I, t' \models \alpha$ for all $t' \in \min_{\prec}(t)$. By our assumption, we have $I, t \models \widetilde{\Diamond}\neg\alpha$. Thus, there is a time point $t' \in \min_{\prec}(t)$ such that $I, t' \models \neg\alpha$. This contradicts with the above fact that $I, t' \models \alpha$ for all $t' \in \min_{\prec}(t)$. Thus, $I, t \models \neg \widetilde{\Diamond}\neg\alpha$ and therefore we conclude that $\models \widetilde{\Box}\alpha \rightarrow \neg \widetilde{\Diamond}\neg\alpha$.

For the if part, we assume that $I, t \models \neg \widetilde{\Diamond}\neg\alpha$. Going back to the semantics of the operator $\widetilde{\Diamond}$, if $I, t \models \widetilde{\Diamond}\neg\alpha$, then there is a $t' \in \min_{\prec}(t)$ such that $I, t \models \neg\alpha$. Therefore, $I, t \models \neg \widetilde{\Diamond}\neg\alpha$ means that there is no $t' \in \min_{\prec}(t)$ where $I, t' \models \neg\alpha$. Hence, for all $t' \in \min_{\prec}(t)$, we have $I, t' \not\models \neg\alpha$, and consequently $I, t' \models \alpha$. We conclude that $I, t \models \widetilde{\Box}\alpha$ and therefore $\models \neg \widetilde{\Diamond}\neg\alpha \rightarrow \widetilde{\Box}\alpha$. □

Analogously as for the classical modalities, we have a duality between the $\widetilde{\Box}$ and $\widetilde{\Diamond}$ operators. The validity $\models \widetilde{\Diamond}\alpha \leftrightarrow \neg \widetilde{\Box}\neg\alpha$ is also true.

Proposition 6.2. *Let α be a well-formed sentence in $\mathcal{L}^{\sim}$. We have:*

$$\models \Box\alpha \rightarrow \widetilde{\Box}\alpha \text{ and } \models \widetilde{\Diamond}\alpha \rightarrow \Diamond\alpha$$

Proof. We take an arbitrary $I = (V, \prec) \in \mathfrak{I}$, $\alpha \in \mathcal{L}^{\sim}$ and $t \in \mathbb{N}$.

- We assume that $I, t \models \Box\alpha$. Then, we have $I, t' \models \alpha$ for all $t' \in [t, \infty[$. Moreover, since $\min_{\prec}(t) \subseteq [t, \infty[$, we have $I, t' \models \alpha$ for all $t' \in \min_{\prec}(t)$. Therefore, we have $I, t \models \widetilde{\Box}\alpha$. We conclude that $\models \Box\alpha \rightarrow \widetilde{\Box}\alpha$.
- We assume that $I, t \models \widetilde{\Diamond}\alpha$. Then, there is a $t' \in \min_{\prec}(t)$ such that $I, t \models \alpha$. Since $\min_{\prec}(t) \subseteq [t, \infty[$ and $t' \in \min_{\prec}(t)$, there is $t' \in [t, \infty[$ such that $I, t' \models \alpha$. Therefore, we have $I, t \models \Diamond\alpha$. We conclude that $\models \widetilde{\Diamond}\alpha \rightarrow \Diamond\alpha$.

□

Proposition 6.2 states that if a statement holds in all of future time points of any given point of time t, it holds on all *preferred future* time points. As intended, this property establishes the defeasible always as "weaker" than the classical always. It can commonly be accepted since the set of all preferred future states are in the future. This is why we named $\widetilde{\Box}$ *defeasible always.* On the other hand, we see that $\widetilde{\Diamond}$ is "stronger" than classical eventually, the statement within $\widetilde{\Diamond}$ holds at a preferable future.

Next, we discuss the axioms that hold for classical modalities ($\Box, \Diamond$) and compare them with defeasible modalities ($\widetilde{\Box}, \widetilde{\Diamond}$). In the case of classical modalities, the distributivity axiom (K) $\models \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$, the reflexivity axiom (T) $\models \Box\alpha \rightarrow \alpha$ and the transitivity axiom (4) $\models \Box\alpha \rightarrow \Box\Box\alpha$ are valid (1). As for defeasible modalities, we have the following:

Proposition 6.3 (Axiom $\tilde{K}$)**.** *Let $\alpha, \beta \in \mathcal{L}^{\sim}$. We have:*

$$(\tilde{K}) \quad \models \widetilde{\Box}(\alpha \rightarrow \beta) \rightarrow (\widetilde{\Box}\alpha \rightarrow \widetilde{\Box}\beta)$$

Proof. We take an arbitrary $I = (V, \prec) \in \mathfrak{I}$, $\alpha, \beta \in \mathcal{L}^{\sim}$ and $t \in \mathbb{N}$. We assume that $I, t \models \widetilde{\Box}(\alpha \rightarrow \beta)$ and suppose that $I, t \not\models \widetilde{\Box}\alpha \rightarrow \widetilde{\Box}\beta$. Since $I, t \models \widetilde{\Box}(\alpha \rightarrow \beta)$, we have (1) $I, t' \models \alpha \rightarrow \beta$ for all $t' \in \min_{\prec}(t)$. Going back to the supposition, if $I, t \not\models \widetilde{\Box}\alpha \rightarrow \widetilde{\Box}\beta$, then $I, t \models \widetilde{\Box}\alpha$ and $I, t \models \neg \widetilde{\Box}\beta$. Using duality, we obtain $I, t \models \widetilde{\Box}\alpha$ and $I, t \models \widetilde{\Diamond}\neg\beta$. Since $I, t \models \widetilde{\Diamond}\neg\beta$, there is a $t'' \in \min_{\prec}(t)$ where $I, t'' \models \neg\beta$. Moreover, since $t'' \in \min_{\prec}(t)$ and $I, t \models \widetilde{\Box}\alpha$, we have $I, t'' \models \alpha$. Therefore, we have $I, t'' \models \alpha \wedge \neg\beta$. Thanks to De Morgan's law, we obtain $I, t'' \models \neg(\alpha \rightarrow \beta)$. The result of the supposition conflicts with the assumption (1), as t'' is also in $\min_{\prec}(t)$ and thus $\alpha \rightarrow \beta$ is true at t''. Therefore, we have $I, t \models \widetilde{\Box}\alpha \rightarrow \widetilde{\Box}\beta$. We conclude that $\models \widetilde{\Box}(\alpha \rightarrow \beta) \rightarrow (\widetilde{\Box}\alpha \rightarrow \widetilde{\Box}\beta)$. □

The axiom of distributivity (K) can be stated in terms of our defeasible temporal operators. We can also verify the validity of these two statements $\models \widetilde{\Box}(\alpha \wedge \beta) \leftrightarrow (\widetilde{\Box}\alpha \wedge \widetilde{\Box}\beta)$ and $\models (\widetilde{\Box}\alpha \vee \widetilde{\Box}\beta) \rightarrow \widetilde{\Box}(\alpha \vee \beta)$.

Proof.

- $\models \widetilde{\Box}(\alpha \wedge \beta) \leftrightarrow (\widetilde{\Box}\alpha \wedge \widetilde{\Box}\beta)$. We take an arbitrary $I = (V, \prec) \in \mathfrak{I}$, $\alpha, \beta \in \mathcal{L}^{\sim}$ and $t \in \mathbb{N}$. For the if part, we assume that $I, t \models \widetilde{\Box}\alpha \wedge \widetilde{\Box}\beta$. For all $t' \in \min_{\prec}(t)$, we have $I, t' \models \alpha$ and $I, t' \models \beta$. Therefore, we have $I, t' \models \alpha \wedge \beta$ and thus $I, t \models \widetilde{\Box}(\alpha \wedge \beta)$. For the only if part, we assume that $I, t \models \widetilde{\Box}(\alpha \wedge \beta)$. For all $t' \in \min_{\prec}(t)$, we have $I, t' \models \alpha \wedge \beta$. Then, for all $t' \in \min_{\prec}(t)$, we have $I, t' \models \alpha$ and also $I, t' \models \beta$. Thus, we have $I, t \models \widetilde{\Box}\alpha$ and $I, t \models \widetilde{\Box}\beta$ and therefore $I, t \models \widetilde{\Box}\alpha \wedge \widetilde{\Box}\beta$.
- $\models (\widetilde{\Box}\alpha \vee \widetilde{\Box}\beta) \rightarrow \widetilde{\Box}(\alpha \vee \beta)$. We take an arbitrary $I = (V, \prec) \in \mathfrak{I}$, $\alpha, \beta \in \mathcal{L}^{\sim}$ and $t \in \mathbb{N}$ such that $I, t \models \widetilde{\Box}\alpha \vee \widetilde{\Box}\beta$. We have $I, t \models \widetilde{\Box}\alpha$ or $I, t \models \widetilde{\Box}\beta$. We assume that $I, t \models \widetilde{\Box}\alpha$. It follows that $I, t' \models \alpha$ for all $t' \in \min_{\prec}(t)$. Then, we have $I, t' \models \alpha \vee \beta$ for all $t' \in \min_{\prec}(t)$. Therefore, we have $\widetilde{\Box}(\alpha \vee \beta)$.

□

Similarly to the operator $\Box$, the validity $\models \widetilde{\Box}(\alpha \vee \beta) \rightarrow (\widetilde{\Box}\alpha \vee \widetilde{\Box}\beta)$ is not true. Assume that a preferential interpretation I satisfies $\widetilde{\Box}(\alpha \vee \beta)$ at t. This means that for all $t' \in \min_{\prec}(t)$, either $I, t' \models \alpha$ or $I, t' \models \beta$. Let say that α is true for all $t' \in \min_{\prec}(t)$ except for one time point t'' which satisfies β instead. In this case, neither $\widetilde{\Box}\alpha$ nor $\widetilde{\Box}\beta$ are true in t.

Proposition 6.4 (Reflexivity)**.** *Let $\alpha \in \mathcal{L}^{\sim}$. We have:*

$$(\tilde{T}) \qquad \not\models \widetilde{\Box}\alpha \rightarrow \alpha$$

The reflexivity axiom $(\tilde{T})$ for the classical operators does not hold in the case of defeasible modalities. We can easily find an interpretation $I = (V, \prec)$ where $I, t \not\models \widetilde{\Box}\alpha \rightarrow \alpha$. Indeed, since we can have $t \notin \min_{\prec}(t)$ for a temporal point t, we can have $I, t \models \widetilde{\Box}\alpha$ and $I, t \models \neg\alpha$. Case in point on the interpretation in Figure 2, the set of preferred future time points relative to 3 is $\min_{\prec}(3) = [4, \infty[$. We can see that $I, t' \models x_2 \rightarrow y_3$ for all $t' \in \min_{\prec}(3)$ and therefore $I, 3 \models \widetilde{\Box}(x_2 \rightarrow y_3)$. However, we have $I, 3 \not\models x_2 \rightarrow y_3$.

Proposition 6.5 (Transitivity)**.** *Let $\alpha \in \mathcal{L}^{\sim}$. We have:*

$$(\tilde{4}) \qquad \not\models \widetilde{\Box}\alpha \rightarrow \widetilde{\Box}\,\widetilde{\Box}\alpha$$

It is worth to point out that the set of future preferred time points changes dynamically as we move forward in time. Given three time points $t_1 \leq t_2 \leq t_3$, $t_3 \notin \min_{\prec}(t_1)$,

$t_2 \in \min_{\prec}(t_1)$ whilst $t_3 \in \min_{\prec}(t_2)$ could be true in some cases. Hence, if $I, t_1 \models \Box\!\!\!\sim\alpha$ does not imply that $I, t_2 \models \Box\!\!\!\sim\alpha$.

Example 6.6. Consider a preferential interpretation $I = (V, \prec)$ and five time points $t_0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$ such that $\prec = \{(t_1, t_0), (t_1, t_3)\}$, $I, t_i \models \alpha$ for each $i \in \{1, 2, 4\}$ and t_3 does not satisfy α. We can see that $\min_{\prec}(t_0) = \{t_1, t_2, t_4\}$. Hence, we have $I, t_0 \models \Box\!\!\!\sim\alpha$, since $I, t_i \models \alpha$ for all $t_i \in \min_{\prec}(t_0)$.

Moving on to t_2, we have $\min_{\prec}(t_2) = \{t_3, t_4\}$. In this case, $I, t_2 \not\models \Box\!\!\!\sim\alpha$, since $t_3 \in \min_{\prec}(t_2)$ and $I, t_3 \not\models \alpha$. Moreover, since $t_2 \in \min_{\prec}(t_0)$ and $I, t_2 \not\models \Box\!\!\!\sim\alpha$, then we have $I, t_0 \not\models \Box\!\!\!\sim\Box\!\!\!\sim\alpha$.

Therefore, the transitivity axiom $(\tilde{4})$ does not hold in the case of defeasible modalities. On the other hand, given those three time points, $t_3 \notin \min_{\prec}(t_2)$ implies that $t_3 \notin \min_{\prec}(t_1)$.

We argue that since defeasible modalities are non-monotonic in nature, the reflexivity and transitivity axioms for these type of modalities do not hold. In the case of classical modalities, by combining both (T) and (4) axioms, we obtain the validity $\models \Box\Box\alpha \leftrightarrow \Box\alpha$. Using duality, we also obtain $\models \Diamond\Diamond\alpha \leftrightarrow \Diamond\alpha$. And as discussed in Proposition 6.4 and 6.5, the two aforementioned validities are false in the case of defeasible modalities, i.e., $\not\models \Box\!\!\!\sim\Box\!\!\!\sim\alpha \leftrightarrow \Box\!\!\!\sim\alpha$ and $\not\models \Diamond\!\!\!\sim\Diamond\!\!\!\sim\alpha \leftrightarrow \Diamond\!\!\!\sim\alpha$. Therefore, there is no collapsing when it comes to defeasible temporal operators.

7. $LTL^{\sim}$ sub-languages

In this paper, we will focus on two subsets of the language, namely, $\mathcal{L}_1$ and $\mathcal{L}_2$. In the sub-language $\mathcal{L}_1$, we omit $\mathcal{U}$ and $\Box\!\!\!\sim$ from the set of modalities. Moreover, only Boolean sentences are allowed within the scope of $\Box$ sentences. In the second subset $\mathcal{L}_2$, the language contains only Boolean connectives, the two defeasible operators $\Box\!\!\!\sim$, $\Diamond\!\!\!\sim$ and their classical counterparts.

7.1. *The fragment* $\mathcal{L}_1$

The set of operators consists of $\wedge, \vee, \Diamond, \Box, \bigcirc, \Diamond\!\!\!\sim$. We shall assume that sentences in $\mathcal{L}_1$ are in negation normal form, which means that negation is only applied to atomic propositions. Furthermore, only Boolean connectors are allowed within the scope of $\Box$ sentences. Temporal operators, classical or non-monotonic, are not permitted in the scope of $\Box$ sentences.

In what follows, we describe well formed sentences of $\mathcal{L}_1$. In order to do that, we define first the set of Boolean sentences $\mathcal{L}_{bool}$. Let $p \in \mathcal{P}$, sentences $\alpha_{bool} \in \mathcal{L}_{bool}$ are defined recursively as such:

$$\alpha_{bool} ::= p \mid \neg p \mid \alpha_{bool} \wedge \alpha_{bool} \mid \alpha_{bool} \vee \alpha_{bool}$$

Next, let $\alpha_{bool} \in \mathcal{L}_{bool}$, sentences in $\mathcal{L}_1$ are recursively defined as such:

$$\alpha ::= \alpha_{bool} \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Diamond\alpha \mid \Box\alpha_{bool} \mid \bigcirc\alpha \mid \Diamond\!\!\!\sim\alpha$$

While the expressivity of $\mathcal{L}_1$ is restricted, we can express a variety of properties using this language. For instance, we can check for the pertinent liveness property $\tilde{\Diamond}\alpha$, liveness property $\Diamond\alpha$ and the persistence property $\Diamond\Box\alpha_{bool}$. We can also express another version of the defeasible persistence property $\tilde{\Diamond}\Box\alpha_{bool}$ (after a normal time point, α_{bool} holds in all future time points). Nevertheless, defeasible safety $\tilde{\Box}\alpha$ is not allowed and only safety of Boolean properties is allowed $\Box\alpha_{bool}$.

Example 7.1. Here are some examples of well-formed sentences in $\mathcal{L}_1$. Let $p, q \in \mathcal{P}$:

$$p, \quad \neg p, \quad \Box(p \wedge q) \rightarrow \Diamond p, \quad \tilde{\Diamond}\Box p, \quad \bigcirc\Box(p \vee q)$$

The following sentences are not well-formed sentences in $\mathcal{L}_1$:

$$\tilde{\Box}p, \quad \Box\Diamond(p \wedge q), \quad \Box\,\tilde{\Diamond}p$$

7.2. The fragment $\mathcal{L}_2$

The second fragment $\mathcal{L}_2$ is a sub-language of $\mathcal{L}^{\sim}$ on which only Boolean connectives and the temporal operators $\Box$, $\tilde{\Box}$, $\Diamond$, $\tilde{\Diamond}$ (the operators $\bigcirc$, $\mathcal{U}$ are omitted) are allowed as connectives. Sentences in $\mathcal{L}_2$ are recursively defined as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \tilde{\Box}\alpha \mid \tilde{\Diamond}\alpha$$

The fragment $\mathcal{L}_2$ is more expressive than $\mathcal{L}_1$. All classical and defeasible properties that are discussed in Sections 2.1 and 4 can be expressed using this fragment. With the absence of $\bigcirc$, the inductive form of both $\Box$ and $\Diamond$ cannot be expressed, i.e., $\models \Box\alpha \leftrightarrow \alpha \wedge \bigcirc\Box\alpha$ and $\models \Diamond\alpha \leftrightarrow \alpha \vee \bigcirc\Diamond\alpha$.

Example 7.2. Here are some examples of well-formed sentences in $\mathcal{L}_2$. Let $p, q \in \mathcal{P}$:

$$p, \quad \neg p, \quad \Box(p \wedge q) \rightarrow \Diamond p, \quad \tilde{\Diamond}\Box p, \quad \Box\,\tilde{\Diamond}(p \vee q), \quad \tilde{\Box}p, \quad \tilde{\Box}\Diamond p, \quad \tilde{\Box}\,\tilde{\Diamond}(p \rightarrow q)$$

The following sentences are not well-formed sentences in $\mathcal{L}_2$:

$$\bigcirc p, \quad p\,\mathcal{U}\,q, \quad \bigcirc\Box(p \wedge q)$$

We based the syntax of $\mathcal{L}_1$ on $L(\Diamond)$, and the syntax of $\mathcal{L}_2$ on $L_{NNF}(\Diamond, \bigcirc)$ in Sistla and Clarke's work (19). In regards to the fragment $\mathcal{L}_1$, sentences in $\mathcal{L}_1$ follow a similar pattern to the $L_{NNF}(\Diamond, \bigcirc)$ fragments, with the addition of $\tilde{\Diamond}$ and allowing $\Box$ sentences only when they have α_{bool} sentences in their scope. For the fragment $\mathcal{L}_2$, we add our defeasible temporal operators $\tilde{\Box}$, $\tilde{\Diamond}$ to the fragment $L(\Diamond)$. In the upcoming sections, we discuss the satisfiability problem of sentences in these two fragments.

With the $\mathcal{L}^{\sim}$ language and preferential temporal interpretations defined, we present an analysis of the satisfiability of $\mathcal{L}^{\sim}$ sentences. The algorithmic problem is as follows: Given an input sentence $\alpha \in \mathcal{L}^{\sim}$, decide whether α is preferentially satisfiable. Sistla and Clarke (19) provide, depending on the fragment of $\mathcal{L}$ language, structures that are useful to prove the bounded model property. Then, they lay out the procedures for checking the satisfiability of the sentence within each of these fragments. In order to establish computational properties about the satisfiability problem in LTL extended with defeasible operators such as those we have considered so far. We introduce structures and $LTL^{\sim}$ fragments inspired by the approach put forward by Sistla and Clarke (19). A part of this work was published in Chafik et al. (5).

The upcoming sections are divided into four parts: we shall discuss in Section 8 an interesting sub-class of $LTL^{\sim}$ interpretations that is useful for establishing the bounded model property for a part of the language. Next, we investigate in Section 9 Sistla and Clarke's notations for preferential temporal structures. We proceed then to establish the bounded model property for two of $\mathcal{L}^{\sim}$ fragments, namely $\mathcal{L}_1$ (Section 10) and $\mathcal{L}_2$ (Section 12). Finally, we provide a procedure for checking the satisfiability of sentences within these fragments (Section 11 for $\mathcal{L}_1$ sentences and Section 13 for $\mathcal{L}_2$ sentences).

8. State-dependent preferential interpretations

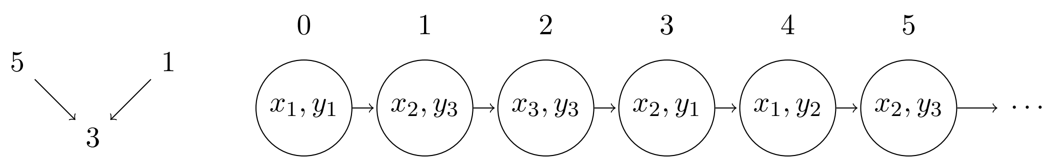
The complexity of the satisfiability problem for LTL has been investigated by Sistla and Clarke (19). Since temporal structures are infinite by nature, finite representations of these structures were put in place in order to check the computational properties of LTL. In the case of $LTL^{\sim}$, the preferential component of $\mathfrak{I}$ interpretations could also be infinite. That is why in the study of the satisfiability problem of $LTL^{\sim}$, we define a well-behaved ordering relation $\prec$. In this section, we introduce a subclass of $\mathfrak{I}$-interpretations called state-dependent interpretations.

Definition 8.1 (State-dependent preferential interpretations)**.** Let $I = (V, \prec) \in \mathfrak{I}$. I is a state-dependent preferential interpretation iff for every $i, j, i', j' \in \mathbb{N}$, if $V(i') = V(i)$ and $V(j') = V(j)$, then $(i, j) \in \prec$ iff $(i', j') \in \prec$.

The notation $\mathfrak{I}^{sd}$ denotes the set of all state-dependent interpretations.

Example 8.2. Let take the preferential temporal interpretation represented of the second run (see Figure 3 for a reminder).

Recall that, time points where $x = 2$ and $y = 3$ are more preferred than time points where $x = 2$ and $y = 1$. In the previous interpretation $I = (V, \prec)$, we had $\prec := \{(5, 3), (1, 3)\}$. Note that for all $t > 5$, we have $V(t) = \{x_2, y_3\}$. Now that if all time points with valuations 1 and 5 are also more preferred than 3, we can use a state-dependent interpretation $I' \in \mathfrak{I}^{sd}$ to represent this case. The interpretation $I' = (V', \prec')$ has the same valuation function as the valuation function V in I. In addition, for all $(t, t') \in \mathbb{N}^2$ such that $V'(t) = \{x_2, y_3\}$ and $V'(t') = \{x_2, y_1\}$, we have $(t, t') \in \prec'$. In other words, the relation $\prec'$ can be defined as such: $\prec' = \{(1, 3), (5, 3)\} \cup \{(t, t') \in \mathbb{N}^2 \mid V'(t) = \{x_2, y_3\}$ and $V'(t') = \{x_2, y_1\}\}$.



(a) Preference relation $\prec$ (b) For $t > 5$, $V(t) = V(5) = \{x_2, y_3\}$

Figure 3.: Preferential temporal interpretation $I = (V, \prec)$

The intuition behind setting up this restriction is to have a more compact form of expressing the ordering relation over time points. In general, time points that have the same valuations are identical with regards to $\prec$, they express the same normality towards other time points. Moreover, we have an interesting property that does not hold in the general case.

Proposition 8.3. *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i, i', j, j' \in \mathbb{N}$ s.t. $i \leq i'$, $i' \leq j'$ and $j \in \min_{\prec}(i)$. If $V(j) = V(j')$, then $j' \in \min_{\prec}(i')$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let i, j, i', j' be four time points s.t. $i \leq i'$, $i' \leq j'$ and $j \in \min_{\prec}(i)$. We assume that $V(j) = V(j')$ and we suppose that $j' \notin \min_{\prec}(i')$. Following our supposition, $j' \notin \min_{\prec}(i')$ means that there exists $k \in [i', \infty[$ where $(k, j') \in \prec$. From Definition 8.1, if $(k, j') \in \prec$ and $V(j) = V(j')$, then $(k, j) \in \prec$. Since $(k, j) \in \prec$, we have $j \notin \min_{\prec}(i)$. This conflicts with our assumption of $j \in \min_{\prec}(i)$. We conclude that if $V(j) = V(j')$ then $j' \in \min_{\prec}(i')$. $\square$

Proposition 8.3 states that whenever $j \in \min_{\prec}(i)$, i.e., j is a preferred future time point of i, then all $j' \geq i$ with the same valuation as j are preferred futures of all time points $i \leq i' \leq j'$. This property is specific to the class of state-dependent interpretations. We add another property that holds on all interpretations $I \in \mathfrak{I}$.

Proposition 8.4. *Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, j \in \mathbb{N}$ s.t. $j \in \min_{\prec}(i)$. For all $i \leq i' \leq j$, we have $j \in \min_{\prec}(i')$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$ and let $i, i', j \in \mathbb{N}$ s.t. $j \in \min_{\prec}(i)$ and $i \leq i' \leq j$. Since $j \in \min_{\prec}(i)$, there is no $j' \in [i, \infty[$ s.t. $(j', j) \in \prec$. Moreover, we have $i \leq i'$, we conclude that there is no $j' \in [i', \infty[$ s.t. $(j', j) \in \prec$. Therefore, we have $j \in \min_{\prec}(i')$. $\square$

In the case of $\mathfrak{I}$-interpretations, when a time point j is a preferred time point of i, then the time point j remains a preferred time point of all time points between i and j. State-dependent interpretations are going to be used as the de facto interpretations for the fragment in $\mathcal{L}_2$. We shall study them in more depth in Sections 12 and 13.

9. Useful representations of preferential structures

Throughout this work, the term temporal sequence, or sequence in short, will denote a sequence of integer numbers in their natural order. A sequence represents a set of time points. Sequences can also be finite or infinite. In what follows, we define formally the notion of sub-sequences.

Definition 9.1 (Sub-sequence)**.** Let N, N' be two sequences of natural numbers. N' is a subsequence of N (written as $N' \subseteq N$) iff for all $i \in N'$, we have $i \in N$.

We introduce pseudo-interpretations next. A pseudo-interpretation I^N over a sequence N is the restriction of the valuation and the ordering relation of the interpretation $I = (V, \prec)$ to time points of N.

Definition 9.2 (Pseudo-interpretation over N)**.** Let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence of natural numbers. The pseudo-interpretation over N is the pair $I^N \stackrel{\text{def}}{=} (V^N, \prec^N)$ where:

- $V^N : N \longrightarrow 2^{\mathcal{P}}$ is a valuation function over N, where for all $i \in N$, we have

$V^N(i) = V(i)$,
- $\prec^N \subseteq N \times N$, where for all $(i,j) \in N^2$, we have $(i,j) \in \prec^N$ iff $(i,j) \in \prec$.

With pseudo-interpretations, we can check the truth values of sentences within sequences of the starting interpretation I. The truth values of $\mathcal{L}^{\sim}$ sentences in pseudo-interpretations are defined in a similar fashion as for preferential temporal interpretations. Let $t, t' \in N$, with $\models_{\mathscr{P}}$ we denote the truth values of sentences in a pseudo-interpretation.

- $I^N, t \models_{\mathscr{P}} p$ if $p \in V^N(t)$;
- $I^N, t \models_{\mathscr{P}} \neg\alpha$ if $I^N, t \not\models_{\mathscr{P}} \alpha$;
- $I^N, t \models_{\mathscr{P}} \alpha \wedge \beta$ if $I^N, t \models_{\mathscr{P}} \alpha$ and $I^N, t \models_{\mathscr{P}} \beta$;
- $I^N, t \models_{\mathscr{P}} \alpha \vee \beta$ if $I^N, t \models_{\mathscr{P}} \alpha$ or $I^N, t \models_{\mathscr{P}} \beta$;
- $I^N, t \models_{\mathscr{P}} \Box\alpha$ if $I^N, t' \models_{\mathscr{P}} \alpha$ for all $t' \in N$ s.t. $t' \geq t$;
- $I^N, t \models_{\mathscr{P}} \Diamond\alpha$ if $I^N, t' \models_{\mathscr{P}} \alpha$ for some $t' \in N$ s.t. $t' \geq t$;
- $I^N, t \models_{\mathscr{P}} \bigcirc\alpha$ if we have $t+1 \in N$ and $I^N, t+1 \models_{\mathscr{P}} \alpha$;
- $I^N, t \models_{\mathscr{P}} \boxminus\alpha$ if for all $t' \in N$ s.t. $t' \in \min_{\prec^N}(t)$, we have $I^N, t' \models_{\mathscr{P}} \alpha$;
- $I^N, t \models_{\mathscr{P}} \Diamond\!\!\!\sim\alpha$ if $I^N, t' \models_{\mathscr{P}} \alpha$ for some $t' \in N$ s.t. $t' \in \min_{\prec^N}(t)$.

Another observation made by Sistla & Clarke in the case of finite sets of atomic proposition $\mathcal{P}$ is that in every LTL interpretation, there is a time point t after which every t-successor's valuation occurs infinitely many times. This is an obvious consequence of having an infinite set of time points and a finite number of possible valuations. That is the case also for $LTL^{\sim}$ interpretations.

Lemma 9.3. *Let $I = (V, \prec) \in \mathfrak{I}$. There exists $t \in \mathbb{N}$ s.t. for all $l \in [t, \infty[$, there is a $k > l$ where $V(l) = V(k)$.*

Definition 9.4. For an interpretation $I \in \mathfrak{I}$, we denote the first time point where the condition set in Lemma 9.3 is satisfied by $\mathfrak{t}_I$.

With the delimiter $\mathfrak{t}_I$ defined, we can split each temporal structure into two intervals: an initial and a final part.

Definition 9.5. Let $I = (V, \prec) \in \mathfrak{I}$. We define:

- $init(I) \stackrel{\text{def}}{=} [0, \mathfrak{t}_I[$;
- $final(I) \stackrel{\text{def}}{=} [\mathfrak{t}_I, \infty[$;
- $range(I) \stackrel{\text{def}}{=} \{V(i) \mid i \in final(I)\}$;
- $val(I) \stackrel{\text{def}}{=} \{V(i) \mid i \in \mathbb{N}\}$;
- $size(I) \stackrel{\text{def}}{=} length(init(I)) + card(range(I))$, where $length(\cdot)$ denotes the length of a sequence and $card(\cdot)$ set cardinality.

In the size of I, we count the number of time points in the initial part and the number of valuations contained in the final part. In the absence of $\bigcirc$ and $\mathcal{U}$ operators (such is the case of the fragment $\mathcal{L}_2$), the order of time points in final I does not matter (19). In what follows, we show that it is the case if we use $\mathcal{L}_2$ sentences and $\mathfrak{I}^{sd}$ interpretations.

Proposition 9.6. *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i \leq j \leq i' \leq j'$ be time points in $final(I)$ s.t. $V(j) = V(j')$. Then we have $j \in \min_{\prec}(i)$ iff $j' \in \min_{\prec}(i')$.*

Lemma 9.7. *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ be time points of $final(I)$ where $V(i) =$*

$V(i')$. *Then for every* $\alpha \in \mathcal{L}_2$, *we have* $I, i \models \alpha$ *iff* $I, i' \models \alpha$.

What we have in Lemma 9.7 is that given an interpretation $I \in \mathfrak{I}^{sd}$, points of time in *final*(I) having the same valuations satisfy exactly the same sentences.

Definition 9.8 (Faithful Interpretations)**.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two interpretations over the same set of atomic propositions $\mathcal{P}$. We say that I, I' are faithful interpretations if $val(I) = val(I')$ and, for all $i, j, i', j' \in \mathbb{N}$ s.t. $V'(i') = V(i)$ and $V'(j') = V(j)$, we have $(i, j) \in \prec$ iff $(i', j') \in \prec'$.

Throughout this paper, we write $init(I) \doteq init(I')$ as shorthand for the condition that states: $length(init(I)) = length(init(I'))$ and for each $i \in init(I)$ we have $V(i) = V'(i)$.

Lemma 9.9. *Let* $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ *be two faithful interpretations over* $\mathcal{P}$ *such that* $V'(0) = V(0)$ *(in case* $init(I)$ *is empty),* $init(I) \doteq init(I')$*, and* $range(I) = range(I')$. *Then for all* $\alpha \in \mathcal{L}_2$, *we have that* $I, 0 \models \alpha$ *iff* $I', 0 \models \alpha$.

In the case of an empty initial part, we need to make sure that both of the interpretations start at the same temporal state $V(0)$. Hence, we add the constraint $V'(0) = V(0)$ when $init(I)$ is empty. Lemma 9.9 implies that the ordering of time points in *final*$(\cdot)$ does not matter, and what matters is the *range*$(\cdot)$ of valuations contained within it. It is worth to mention that Lemmas 9.7 and 9.9 hold only in $\mathfrak{I}^{sd}$ interpretations and they are not always true in the general case.

Sistla & Clarke (19) introduced sequences that display a certain behaviour called *acceptable sequences*. We extend the notion of acceptable sequences for preferential temporal interpretations in $\mathfrak{I}$ as follows:

Definition 9.10 (Acceptable sequence w.r.t. I)**.** Let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence of temporal time points. N is an acceptable sequence w.r.t. I iff for all $i, j \in final(I)$ s.t. $V(i) = V(j)$, we have $i \in N$ iff $j \in N$.

The particularity we are looking for is that any picked time point in *init*$(\cdot)$ (resp. *final*$(\cdot)$) will remain in the initial (resp. final) part of the new pseudo-interpretation. It is worth pointing out that an acceptable sequence w.r.t. a preferential temporal interpretation can be either finite or infinite. Moreover, $\mathbb{N}$ is an acceptable sequence w.r.t. any interpretation $I \in \mathfrak{I}$. The purpose behind the notion of acceptable sequence is to build new interpretations starting from an $LTL^{\sim}$ interpretation.

Given N an acceptable sequence w.r.t. I, if N has a time point t in *final*(I), then all time points t' that have the same valuation as t must be in N. Thus, we have an infinite sequence of time points in N. As such, we can define an initial part and a final part, in a similar way as $LTL^{\sim}$ interpretations. We let $init(I, N)$ be the largest subsequence of N that is a subsequence of $init(I)$. Note that if N does not contain any time point of *final*(I), then N is finite. Also, an empty sequence, by definition, is an acceptable sequence w.r.t. I.

We now define the notions *init*$(\cdot)$, *final*$(\cdot)$, *range*$(\cdot)$, and *size*$(\cdot)$ for acceptable sequences.

Definition 9.11. Let $I = (V, \prec) \in \mathfrak{I}$, and let N be an acceptable sequence w.r.t. I. We define the following:

$$\begin{array}{lcl}
\mathit{init}(I, N) & \stackrel{\text{def}}{=} & N \cap \mathit{init}(I); \\
\mathit{final}(I, N) & \stackrel{\text{def}}{=} & N \setminus \mathit{init}(I, N); \\
\mathit{range}(I, N) & \stackrel{\text{def}}{=} & \{V(t) \mid t \in \mathit{final}(I, N)\}; \\
\mathit{val}(I, N) & \stackrel{\text{def}}{=} & \{V(t) \mid t \in N\}; \\
\mathit{size}(I, N) & \stackrel{\text{def}}{=} & \mathit{length}(\mathit{init}(I, N)) + \mathit{card}(\mathit{range}(I, N)).
\end{array}$$

It is worth mentioning that the definition of $\mathit{size}(\cdot)$ is different between acceptable sequences and normal (non-acceptable) sequences. The reason behind it that normal sequences do not have the any restrictions compared to acceptable sequences. Thus, the initial part of the normal sequence is not necessarily included in the initial part of the interpretation, the same goes for the final part. In the case of a *finite* normal sequence N, the size of I^N is defined by $\mathit{size}(I, N) \stackrel{\text{def}}{=} \mathit{length}(N)$. Whereas for acceptable sequences, the size of the pseudo-interpretation is the length of the initial part plus the number of distinct valuations in the final part. Thanks to Definition 9.10, given an acceptable sequence w.r.t. I, we have $\mathit{size}(I, N) \leq \mathit{size}(I)$.

Let N_1, N_2 be two sequences of integers. The union of N_1 and N_2, denoted by $N_1 \cup N_2$, is the sequence containing only and all elements of N_1 and N_2. If N_1, N_2 are acceptable sequences, we have the following properties:

Proposition 9.12. *Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I. Then $N_1 \cup N_2$ is an acceptable sequence w.r.t. I s.t. $\mathit{size}(I, N_1 \cup N_2) \leq \mathit{size}(I, N_1) + \mathit{size}(I, N_2)$.*

Proposition 9.13. *Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. If for all distinct $t, t' \in N$, we have $V(t') = V(t)$* only *when both $t, t' \in \mathit{final}(I, N)$, then $\mathit{size}(I, N) \leq 2^{|\mathcal{P}|}$.*

In the upcoming sections, we use sequences to establish the bounded model-property of the fragment $\mathcal{L}_1$ (Section 10) and we use acceptable sequences for the bounded-model property of the fragment $\mathcal{L}_2$ (Section 12).

10. The bounded-model property of the fragment $\mathcal{L}_1$

The first contribution is to establish certain computational properties regarding the satisfiability problem in $\mathcal{L}_1$ (see Section 7.1). Let $\mathcal{P}$ be a finite set of atomic propositions. Just as a remainder, sentences in $\mathcal{L}_1$ are recursively defined as follows:

$$\alpha ::= \alpha_{\mathit{bool}} \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Diamond\alpha \mid \Box\alpha_{\mathit{bool}} \mid \bigcirc\alpha \mid \Diamond\!\!\!\!\sim\alpha$$

Where α_{bool} is a sentence that has only Boolean connectives. Next, we discuss the satisfiability of $\mathcal{L}_1$ sentences. Given an $\mathfrak{I}$-satisfiable sentence $\alpha \in \mathcal{L}_1$, there exists an interpretation $I \in \mathfrak{I}$ s.t. $I, 0 \models \alpha$. From I, we can find a finite sequence of integers N s.t. the pseudo-interpretation I^N satisfies α, i.e., $I^N, 0 \models_{\mathscr{P}} \alpha$. Then, we can transform the pseudo-interpretation I^N into an interpretation I' which has the same size and satisfies the sentence α. The first observation we make is that if an interpretation

I satisfies a Boolean sentence $\alpha_{bool} \in \mathcal{L}_{bool}$ at a time point t, then for all pseudo-interpretations I^N over sequences N that contain t, we have $I^N, t \models_{\mathscr{P}} \alpha_{bool}$. We can extend it further and obtain Proposition 10.1.

Proposition 10.1. *Let $\alpha_{bool} \in \mathcal{L}_{bool}$, let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence containing a time point t s.t. $I^N, t \models_{\mathscr{P}} \alpha_{bool}$. Then for all $N' \subseteq N$ containing t, we have $I^{N'}, t \models_{\mathscr{P}} \alpha_{bool}$.*

Proof. Let $\alpha_{bool} \in \mathcal{L}_{bool}$, let $I = (V, \prec) \in \mathfrak{I}$ and N be a sequence containing t s.t. $I^N, t \models_{\mathscr{P}} \alpha_{bool}$. Let N' be a subsequence of N that contains t. We use structural induction based on α_{bool}.

- $\alpha_{bool} = p$. Since $I^N, t \models_{\mathscr{P}} p$, we know that $p \in V^N(t)$ and therefore $p \in V(t)$. On the other hand, since we have $t \in N'$ and $p \in V(t)$, then we have $p \in V^{N'}(t)$. Therefore, we have $I^{N'}, t \models_{\mathscr{P}} p$.
- $\alpha_{bool} = \neg p$. Since $I^N, t \models_{\mathscr{P}} \neg p$, we know that $p \notin V^N(t)$ and therefore $p \notin V(t)$. On the other hand, since we have $t \in N'$ and $p \notin V(t)$, then we have $p \notin V^{N'}(t)$. Therefore, we have $I^{N'}, t \models_{\mathscr{P}} \neg p$.
- $\alpha_{bool} = \alpha_1 \wedge \alpha_2$. We have $I^N, t \models_{\mathscr{P}} \alpha_1 \wedge \alpha_2$, which means $I^N, t \models_{\mathscr{P}} \alpha_1$ and $I^N, t \models_{\mathscr{P}} \alpha_2$. Since N' is a subsequence of N containing t, by the induction hypothesis on α_1 and α_2, we have $I^{N'}, t \models_{\mathscr{P}} \alpha_1$ and $I^{N'}, t \models_{\mathscr{P}} \alpha_2$. Therefore, we have $I^{N'}, t \models_{\mathscr{P}} \alpha_1 \wedge \alpha_2$.
- $\alpha_{bool} = \alpha_1 \vee \alpha_2$. We have $I^N, t \models_{\mathscr{P}} \alpha_1 \vee \alpha_2$, which means either $I^N, t \models_{\mathscr{P}} \alpha_1$ or $I^N, t \models_{\mathscr{P}} \alpha_2$. We suppose that $I^N, t \models_{\mathscr{P}} \alpha_1$. Since N' is a subsequence of N containing t, by the induction hypothesis on α_1, we have $I^{N'}, t \models_{\mathscr{P}} \alpha_1$. Therefore, we have $I^{N'}, t \models_{\mathscr{P}} \alpha_1 \vee \alpha_2$. Same reasoning applies when $I^N, t \models_{\mathscr{P}} \alpha_2$.

$\square$

Next, let $I \in \mathfrak{I}$ be an interpretation, N be a sequence, $\alpha \in \mathcal{L}_1$ and $t \in N$ s.t. $I^N, t \models_{\mathscr{P}} \alpha$. We can show, using structural induction on α, that we can find a finite sequence M that contains t and such that $I^M, t \models_{\mathscr{P}} \alpha$. Moreover, for all sequences $M \subseteq Q \subseteq N$ we have $I^Q, t \models_{\mathscr{P}} \alpha$. We show in the following lemma that $size(I, M) \leq |\alpha|$ ($|\alpha|$ denotes the number of symbols within α).

Lemma 10.2. *Let $\alpha \in \mathcal{L}_1$, $I = (V, \prec) \in \mathfrak{I}$, $N \subseteq \mathbb{N}$ and $t \in N$ s.t. $I^N, t \models_{\mathscr{P}} \alpha$. Then there exists a **finite** sequence M containing t such that:*

(1) $M \subseteq N$;
(2) $size(I, M) \leq |\alpha|$;
(3) for all sequences Q where $M \subseteq Q \subseteq N$, we have $I^Q, t \models_{\mathscr{P}} \alpha$.

Proof. Let $\alpha \in \mathcal{L}_1$, $I = (V, \prec) \in \mathfrak{I}$, $t \in N$ and $N \subseteq \mathbb{N}$ s.t. $I^N, t \models_{\mathscr{P}} \alpha$. We use induction on the structure of α.

- $\alpha = p$. Let $M = (t)$ be a sequence containing only t. Then M is a finite sequence such that:
  (1) since $t \in N$, then $M \subseteq N$; (2) we have $size(I, M) = 1 = |p|$; (3) since $I^N, t \models_{\mathscr{P}} p$. Then we have $p \in V(t)$. Let Q be a sequence s.t. $M \subseteq Q \subseteq N$, we have $t \in Q$. Therefore, we have $p \in V^Q(t)$ and $I^Q, t \models_{\mathscr{P}} p$.
- $\alpha = \neg p$. Let $M = (t)$ be a sequence containing only t. Then M is a finite sequence such that:
  (1) since $t \in N$, then $M \subseteq N$; (2) we have $size(I, M) = 1 \leq |\neg p|$; (3) since

$I^N, t \models_{\mathscr{P}} \neg p$, then we have $p \notin V(t)$. Let Q be a sequence where $M \subseteq Q \subseteq N$, we have $t \in Q$. Therefore, we have $p \notin V^Q(t)$ and $I^Q, t \models_{\mathscr{P}} \neg p$.

- $\alpha = \alpha_1 \wedge \alpha_2$. Since $I^N, t \models_{\mathscr{P}} \alpha_1 \wedge \alpha_2$, we then have $I^N, t \models_{\mathscr{P}} \alpha_1$ and $I^N, t \models_{\mathscr{P}} \alpha_2$. Using the induction hypothesis on α_1, there exists a finite sequence M_1 containing t such that:

  (1) we have $M_1 \subseteq N$; (2) we have $\mathit{size}(I, M_1) \leq |\alpha_1|$; (3) for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models_{\mathscr{P}} \alpha_1$.

  Similarly, using the induction hypothesis on α_2, there exists a finite sequence M_2 such that:

  (1) we have $M_2 \subseteq N$; (2) we have $\mathit{size}(I, M_2) \leq |\alpha_2|$; (3) for all sequences Q where $M_2 \subseteq Q \subseteq N$, we have $I^Q, t \models_{\mathscr{P}} \alpha_2$.

  Let $M = M_1 \cup M_2$. Since M_1 and M_2 contain t, then M is a finite sequence that contains t such that:

  (1) since $M_1 \subseteq N$ and $M_2 \subseteq N$, then we have $M_1 \cup M_2 \subseteq N$; (2) thanks to Proposition 9.12, we have $\mathit{size}(I, M) = \mathit{size}(M_1 \cup M_2) \leq \mathit{size}(I, M_1) + \mathit{size}(I, M_2) \leq |\alpha_1| + |\alpha_2| \leq |\alpha_1 \wedge \alpha_2|$; (3) let $M \subseteq Q \subseteq N$ be a sequence. Since $M_1 \subseteq Q \subseteq N$, then we have $I^Q, t \models_{\mathscr{P}} \alpha_1$. Similarly, since $M_2 \subseteq Q \subseteq N$, then we have $I^Q, t \models_{\mathscr{P}} \alpha_2$. Therefore, we have $I^Q, t \models_{\mathscr{P}} \alpha_1 \wedge \alpha_2$.
- $\alpha = \alpha_1 \vee \alpha_2$. We have either $I^N, t \models_{\mathscr{P}} \alpha_1$ or $I^N, t \models_{\mathscr{P}} \alpha_2$. We suppose that $I^N, t \models_{\mathscr{P}} \alpha_1$. Using the induction hypothesis on α_1, there exists a finite sequence M_1 containing t such that:

  (1) we have $M_1 \subseteq N$; (2) we have $\mathit{size}(I, M_1) \leq |\alpha_1|$; (3) for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models_{\mathscr{P}} \alpha_1$.

  Let $M = M_1$. Since M_1 contains t, then M is a finite sequence that contains t such that:

  (1) we have $M = M_1 \subseteq N$; (2) we have $\mathit{size}(I, M) = \mathit{size}(M_1) \leq |\alpha_1| \leq |\alpha_1 \vee \alpha_2|$; (3) for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t \models_{\mathscr{P}} \alpha_1$. Therefore, $I^Q, t \models_{\mathscr{P}} \alpha_1 \vee \alpha_2$.

  The reasoning is the same when $I^N, t \models_{\mathscr{P}} \alpha_2$.
- $\alpha = \bigcirc\alpha_1$. Since $I^N, t \models_{\mathscr{P}} \bigcirc\alpha_1$, then $t+1 \in N$ and $I^N, t+1 \models_{\mathscr{P}} \alpha_1$. Using the induction hypothesis on α_1, there exists a finite sequence M_1 containing $t+1$ such that:

  (1) we have $M_1 \subseteq N$; (2) we have $\mathit{size}(I, M_1) \leq |\alpha_1|$; (3) for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t+1 \models_{\mathscr{P}} \alpha_1$.

  Let $M = (t) \cup M_1$; then M is a finite sequence containing t such that:

  (1) since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$; (2) thanks to Proposition 9.12, we have $\mathit{size}(I, M) = 1 + \mathit{size}(I, M_1) \leq |\bigcirc\alpha_1|$; (3) let Q be a sequence such that $M \subseteq Q \subseteq N$, we have $t, t+1 \in M$. Since $M_1 \subseteq Q \subseteq N$, then $I^Q, t+1 \models_{\mathscr{P}} \alpha_1$. Therefore, we have $I^Q, t \models_{\mathscr{P}} \bigcirc\alpha_1$.
- $\alpha = \Diamond\alpha_1$. Since $I^N, t \models_{\mathscr{P}} \Diamond\alpha_1$, then there exists $t' \in N$ such that $I^N, t' \models_{\mathscr{P}} \alpha_1$. Using the induction hypothesis on α_1, there exists a finite sequence containing t' such that:

  (1) we have $M_1 \subseteq N$; (2) we have $\mathit{size}(I, M_1) \leq |\alpha_1|$; (3) for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t' \models_{\mathscr{P}} \alpha_1$.

  Let $M = (t) \cup M_1$; then M is a finite sequence containing t such that:

  (1) since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$; (2) thanks to Proposition 9.12, we have $\mathit{size}(I, M) = 1 + \mathit{size}(I, M_1) \leq |\Diamond\alpha_1|$; (3) let Q be a sequence such that $M \subseteq Q \subseteq N$. Then we have $t, t' \in M$. Since $M_1 \subseteq Q \subseteq N$ and $t' \in M_1$, then $I^Q, t' \models_{\mathscr{P}} \alpha_1$. Therefore, we have $I^Q, t \models_{\mathscr{P}} \Diamond\alpha_1$.
- $\alpha = \Diamond\!\!\!\!\cdot\;\alpha_1$. Since $I^N, t \models_{\mathscr{P}} \Diamond\!\!\!\!\cdot\;\alpha_1$, there exists $t' \in N$ s.t. $t' \in \min_{\precsim^N}(t)$. Using the

induction hypothesis on α_1, there exists a finite sequence M_1 containing t' such that:

(1) we have $M_1 \subseteq N$; (2) we have $size(I, M_1) \leq |\alpha_1|$; (3) for all sequences Q where $M_1 \subseteq Q \subseteq N$, we have $I^Q, t' \models_{\mathscr{P}} \alpha_1$.

Let $M = (t) \cup M_1$; then M is a finite sequence containing t such that:

(1) since $M_1 \subseteq N$ and $t \in N$, then we have $M \subseteq N$; (2) thanks to Proposition 9.12, we have $size(I, M) = 1 + size(I, M_1) \leq |\diamondsuit\alpha_1|$; (3) let Q be a sequence such that $M \subseteq Q \subseteq N$. Since we have $t, t' \in M$, $M_1 \subseteq M \subseteq Q \subseteq N$ and $t' \in M_1$, then (i) $I^Q, t' \models_{\mathscr{P}} \alpha_1$.

We suppose that $t' \notin \min_{\precsim^Q}(t)$, there exists $t'' \in Q$ s.t. $(t'', t') \in \prec^Q$. Following this supposition, we have $(t'', t') \in \prec$. Since $t', t'' \in N$, we have $(t'', t') \in \prec^N$, thus $t' \notin \min_{\precsim^N}(t)$. This supposition conflicts with our assumption that $t' \in \min_{\precsim^N}(t)$. Therefore we have (ii) $t' \in \min_{\precsim^Q}(t)$. From (i) and (ii), we conclude that $I^Q, t \models_{\mathscr{P}} \diamondsuit\alpha_1$.

- $\alpha = \Box\alpha_{bool}$. Since $I^N, t \models_{\mathscr{P}} \Box\alpha_{bool}$, we have $I^N, t \models_{\mathscr{P}} \alpha_{bool}$ for all $t' \in N$ s.t. $t' \geq t$. Let $M = (t)$ be a sequence containing only t. Then we have the following:

  (1) we have $M \subseteq N$; (2) we have $size(I, M) = 1 \leq |\Box\alpha_{bool}|$; (3) let $M \subseteq Q \subseteq N$ be a sequence. We need to prove that $I^Q, t \models_{\mathscr{P}} \Box\alpha_{bool}$. Suppose that $I^Q, t \not\models_{\mathscr{P}} \Box\alpha_{bool}$. This means that there exists $t' \in Q$ s.t. $t' \geq t$ and $I^Q, t' \not\models_{\mathscr{P}} \alpha_{bool}$.

  On the other hand, since $t' \in Q$, and $Q \subseteq N$, we have $t' \in N$. We know that $I^N, t \models_{\mathscr{P}} \Box\alpha_{bool}$, and $t' \geq t$, therefore $I^N, t' \models_{\mathscr{P}} \alpha_{bool}$. Thanks to Proposition 10.1, since $\alpha_{bool} \in \mathcal{L}_{bool}$, $t' \in Q \subseteq N$ and $I^N, t' \models_{\mathscr{P}} \alpha_{bool}$, we have $I^Q, t' \models_{\mathscr{P}} \alpha_{bool}$, which raises a contradiction with our assumption. Thus, there is no $t' \in Q$ s.t. $t' \geq t$ and $I^Q, t' \not\models_{\mathscr{P}} \alpha_{bool}$. We conclude that $I^Q, t' \models_{\mathscr{P}} \Box\alpha_{bool}$.

□

The following corollary is a consequence of Lemma 10.2.

Corollary 10.3. *Let $\alpha \in \mathcal{L}_1$ and $I = (V, \prec) \in \mathfrak{I}$ s.t. $I, t \models \alpha$. Then there exists a finite sequence M containing t s.t. $I^M, t \models_{\mathscr{P}} \alpha$ and $size(I, M) \leq |\alpha|$.*

So far, we showed that if we have an interpretation $I \in \mathfrak{I}$ where $I, t \models \alpha$, then we can find a finite sequence M that contains t s.t. $I^M, t \models_{\mathscr{P}} \alpha$. Next, an interpretation $I' \in \mathfrak{I}$ is induced from the pseudo-interpretation I^N which preserves the satisfaction of α. We define formally the construction below.

Definition 10.4 (Interpretation construction)**.** Let $I = (V, \prec) \in \mathfrak{I}$, let $N = (t_0, t_1, t_2, \ldots, t_{n-1})$ where $t_0 < t_1 < t_2 < \cdots < t_{n-1}$ be a finite sequence. The interpretation $I' \stackrel{\text{def}}{=} (V', \prec') \in \mathfrak{I}$ is induced from the pseudo-interpretation $I^N = (V^N, \prec^N)$ as follows:

$$V' : \begin{cases} V'(i) := V^N(t_i) & \text{if } 0 \leq i < n; \\ V'(i) := V^N(t_{n-1}) & \text{otherwise.} \end{cases}$$

And for all $0 \leq i, j < n$ s.t. $(t_i, t_j) \in \prec^N$, we have $(i, j) \in \prec'$.

Let $I^N := (V^N, \prec^N)$ be a pseudo-interpretation and let $I' = (V', \prec')$ be the I^N-induced interpretation. We can see that $size(I') \leq size(I, N)$. The size of the initial part of I' is the sequence N and the final part has one distinct valuation which is the valuation of the last element of the sequence N. We can also see that the truth values of sub-sentences are preserved in the induced interpretation I'. In other words, for every $\alpha \in \mathcal{L}_1$, if $I^N, t_i \models_{\mathscr{P}} \alpha$, then $I', i \models \alpha$.

Theorem 10.5 (Bounded-Model property)**.** *Let $\alpha \in \mathcal{L}_1$ be $\mathfrak{I}$-satisfiable. Then there exists $I = (V, \prec) \in \mathfrak{I}$ s.t. size$(I) \leq |\alpha|$ and $I, 0 \models \alpha$.*

Proof. Let $\alpha \in \mathcal{L}_1$ be $\mathfrak{I}$-satisfiable and let $I = (V, \prec) \in \mathfrak{I}$ where $I, 0 \models \alpha$ is an interpretation that satisfies α. Thanks to Lemma 10.2, since $\mathbb{N}$ is a sequence and $0 \in \mathbb{N}$ s.t. $I, 0 \models \alpha$, then there is a sequence $M \subseteq \mathbb{N}$ containing 0 where $\mathit{size}(I, M) \leq |\alpha|$ and $I^M, 0 \models \alpha$. We obtain I^N-induced interpretation $I' = (V', \prec)$ by changing the labels of M into a sequence of natural numbers and looping the valuation of the last element of M. We can see that $I', 0 \models \alpha$ and $\mathit{size}(I') \leq |\alpha|$.

□

11. The satisfiability problem in $\mathcal{L}_1$

Thanks to Theorem 10.5, if a sentence $\alpha \in \mathcal{L}_1$ is $\mathfrak{I}$-satisfiable, then there exists an interpretation I such that $\mathit{size}(I) \leq |\alpha|$ that satisfies it. Otherwise, if there is no interpretation that satisfies α such that its size is less than the length of α, then the sentence is unsatisfiable. Based on the bounded-model property, we can make a non-deterministic guess for a bounded interpretation and check whether it satisfies the sentence α. Note that the induced $\mathfrak{I}$-interpretations for sentences in $\mathcal{L}_1$ have final parts that consist of only one distinct valuation (see Definition 10.4). Not only that, but the preference relation concerns only time points of the initial part. To this purpose, we introduce a compact structure to represent the bounded interpretations obtained on the last section.

Definition 11.1 (Finite preferential structure)**.** A *finite preferential structure* is a tuple $S = (n, V_S, \prec_S)$ where: n is an integer such that $n \geq 0$ (where n is intended to be the size of the finite sequence); $V_S : [0, n-1] \longrightarrow 2^{\mathcal{P}}$, and $\prec_S \ \subseteq [0, n-1]^2$ is a strict partial order.

We define the size of the structure $\mathit{size}(S) \stackrel{\text{def}}{=} n$. Thanks to these structures, we can build the interpretation $\mathsf{I}(S)$ in the following way:

Definition 11.2. Given a finite preferential structure $S = (n, V_S, \prec_S)$, let $\mathsf{I}(S) \stackrel{\text{def}}{=} (V, \prec)$, $V(t) \stackrel{\text{def}}{=} V_S(t)$, if $t < n$, and $V(t) \stackrel{\text{def}}{=} V_S(n-1)$, otherwise; and $\prec \ \stackrel{\text{def}}{=} \{(t, t') \mid (t, t') \in \prec_S\}$.

Interpretations of Definition 11.2 are $\mathfrak{I}$-interpretations such that:

- there is a time point n after which all time points $t \geq n$ have the same valuation $V(t) = V(n-1)$;
- the preference relation is only on time points within the initial sequence $[0, n-1]$.

Moreover, we have $\mathit{size}(\mathsf{I}(S)) \leq n$, and thus $\mathit{size}(\mathsf{I}(S)) \leq \mathit{size}(S)$. The interpretations of Definition 10.4 $I' = (V', \prec')$ can be viewed as an interpretation $\mathsf{I}(S)$ issued from a finite preferential structure $S = (n, V_S, \prec_S)$. The structure S can be induced such that $n \stackrel{\text{def}}{=} |N|$ (where N is the finite sequence which I' was induced from, and $|N|$ is its cardinality), $V_S \stackrel{\text{def}}{=} V'(t)$ for all $t < n$ and $\prec_S \ \stackrel{\text{def}}{=} \{(t, t') \mid t, t' \in [0, n-1] \text{ and } (t, t') \in \prec'\}$.

We can go from interpretations of Definition 10.4 to finite preferential structures S thanks to the intermediate interpretation $\mathsf{I}(S)$ of Definition 11.2, and go the other way around. We extend also the notion of preferred time points to finite preferential structures S. The formal definition goes as follows: for $t < n$ we have $\min_{\prec_S}(t) \stackrel{\text{def}}{=} \{t' \in [t, n-1] \mid \text{there is no } t'' \in [t, n-1] \text{ with } (t'', t') \in \prec_S\}$. It is easy to show that for every $t, t' \in [0, n-1]$, we have $t' \in \min_{\prec_S}(t)$ iff $t' \in \min_{\prec}(t)$. Finite preferential structures are going to be useful in order to check the satisfiability of the guessed interpretations.

In order to check the satisfiability of $\mathcal{L}_1$ sentences using a finite preferential structure S, we introduce the notion of labelling sets in order to assign a set of sub-sentences of the original sentence α to each element of the sequence $[0, n-1]$ of S. The set of sub-sentences of α is denoted by $\mathit{Sf}(\alpha)$.

Definition 11.3 (Sub-sentences)**.** Let $\alpha \in \mathcal{L}_1$. The set of all sub-sentences of α, denoted by $\mathit{Sf}(\alpha)$, is recursively defined as follows:

- $\mathit{Sf}(p) \stackrel{\text{def}}{=} \{p\}$;
- $\mathit{Sf}(\neg p) \stackrel{\text{def}}{=} \{\neg p\}$;
- $\mathit{Sf}(\alpha_1 \wedge \alpha_2) \stackrel{\text{def}}{=} \mathit{Sf}(\alpha_1) \cup \mathit{Sf}(\alpha_2) \cup \{\alpha_1 \wedge \alpha_2\}$;
- $\mathit{Sf}(\alpha_1 \vee \alpha_2) \stackrel{\text{def}}{=} \mathit{Sf}(\alpha_1) \cup \mathit{Sf}(\alpha_2) \cup \{\alpha_1 \vee \alpha_2\}$;
- $\mathit{Sf}(\Box\alpha_{bool}) \stackrel{\text{def}}{=} \mathit{Sf}(\alpha_{bool}) \cup \{\Box\alpha_{bool}\}$;
- $\mathit{Sf}(\Diamond\alpha_1) \stackrel{\text{def}}{=} \mathit{Sf}(\alpha_1) \cup \{\Diamond\alpha_1\}$;
- $\mathit{Sf}(\bigcirc\alpha_1) \stackrel{\text{def}}{=} \mathit{Sf}(\alpha_1) \cup \{\bigcirc\alpha_1\}$;
- $\mathit{Sf}(\mathrel{\Diamond\!\!\!\!\sim}\alpha_1) \stackrel{\text{def}}{=} \mathit{Sf}(\alpha_1) \cup \{\mathrel{\Diamond\!\!\!\!\sim}\alpha_1\}$.

With a proof by induction, we can show that the cardinality of the set $\mathit{Sf}(\alpha)$ is $|\mathit{Sf}(\alpha)| \leq |\alpha|$. We define for a structure $S = (n, V_S, \prec_S)$ and a sentence $\alpha \in \mathcal{L}_1$, labelling sets $\mathit{lab}^M_\alpha(\cdot)$ which link a set of sub-sentences of α that hold true in each $t \in [0, n-1]$.

Definition 11.4 (Labelling sets)**.** Let $S = (n, V_S, \prec_S)$ be a structure, $\alpha \in \mathcal{L}_1$. The set of sub-sentences of α in a $t \in [0, n-1]$, denoted by $\mathit{lab}^S_\alpha(t)$, is defined as follows:

- $p \in \mathit{lab}^S_\alpha(t)$ iff $p \in V_S(t)$;
- $\neg p \in \mathit{lab}^S_\alpha(t)$ iff $p \notin V_S(t)$;
- $\alpha_1 \wedge \alpha_2 \in \mathit{lab}^S_\alpha(t)$ iff $\alpha_1, \alpha_2 \in \mathit{lab}^S_\alpha(t)$;
- $\alpha_1 \vee \alpha_2 \in \mathit{lab}^S_\alpha(t)$ iff $\alpha_1 \in \mathit{lab}^S_\alpha(t)$ or $\alpha_2 \in \mathit{lab}^S_\alpha(t)$;
- $\Diamond\alpha_1 \in \mathit{lab}^S_\alpha(t)$ iff $\alpha_1 \in \mathit{lab}^S_\alpha(t')$ for some $t' \in [t, n-1]$;
- $\Box\alpha_{bool} \in \mathit{lab}^S_\alpha(t)$ iff $\alpha_{bool} \in \mathit{lab}^S_\alpha(t')$ for all $t' \in [t, n-1]$;
- $\bigcirc\alpha_1 \in \mathit{lab}^S_\alpha(t)$ iff $\alpha_1 \in \mathit{lab}^S_\alpha(t+1)$ and $t+1 \leq n-1$;
- $\mathrel{\Diamond\!\!\!\!\sim}\alpha_1 \in \mathit{lab}^S_\alpha(t)$ iff $\alpha_1 \in \mathit{lab}^S_\alpha(t')$ for some $t' \in \min_{\prec_S}(t)$.

The labelling sets $\mathit{lab}^M_\alpha(\cdot)$ is used to check the satisfiability of the sub-sentences of α in each t in the interval $[0, n-1]$. As mentioned after Definition 11.2, we can represent the bounded interpretations found on the last section. As such, for any given bounded interpretation I', there is a finite structure $S = (n, V_S, \prec_S)$ such that its $\mathsf{I}(S)$ is the same as I' (same valuation for all time points and same preference relation). Given any induced bounded-interpretation I', we show that for every $t \in [0, n-1]$ and every $\alpha_1 \in \mathit{Sf}(\alpha)$ we have $\alpha_1 \in \mathit{lab}^S_\alpha(t)$ iff $I', t \models \alpha_1$. The proof of Lemma 11.5 can be found on the Appendix A.

Lemma 11.5. *Let $\alpha \in \mathcal{L}_1$ be an $\mathfrak{I}$-satisfiable sentence and $I = (V, \prec) \in \mathfrak{I}$ be an interpretation such that $I, 0 \models \alpha$. Let I^N be the pseudo-interpretation of I over the* finite *sequence* N *such that* $I^N, 0 \models_{\mathscr{P}} \alpha$, *and* $I' = (V', \prec')$ *be the induced interpretation from* I^N. *Let* $S = (n, V_S, \prec_S)$ *be the finite preferential structure where* $n = |N|$, $V_S(t) = V'(t)$ *for each* $t \in [0, |N| - 1]$ *and* $\prec_S = \prec'$. *Let* $\mathsf{I}(S) = (V'', \prec'')$ *be the induced interpretation from* S. *We have the following:*

- $\prec'' = \prec'$ *and* $V''(t) = V'(t)$ *for each* $t \in \mathbb{N}$*;*
- *for every* $\alpha_1 \in Sf(\alpha)$, *we have* $\alpha_1 \in lab^S_\alpha(t)$ *iff* $\mathsf{I}(S), t \models \alpha_1$.

The Lemma 11.5 has two interesting consequences. The first one is that we can represent bounded interpretations of Section 10 as finite preferential structures. The second result is labelling sets can be used to check the satisfiability of sub-sentences of α within the finite part of the interpretations. Furthermore, we use Lemma 11.5 and obtain this proposition. In fact, the following proposition is a special case of Lemma 11.5 when $t = 0$.

Corollary 11.6. *Given a finite preferential structure* $S = (n, V_S, \prec_S)$ *and* $\alpha \in \mathcal{L}_1$, *we have* $\mathsf{I}(S), 0 \models \alpha$ *iff* $\alpha \in lab^S_\alpha(0)$.

We describe, in what follows, the algorithm that checks the $\mathfrak{I}$-satisfiability for $\mathcal{L}_1$ sentences. Let α be a sentence in $\mathcal{L}_1$. If α is satisfiable, then there exists an interpretation $I \in \mathfrak{I}$ where $I, 0 \models \alpha$. Thanks to Theorem 10.5, a new interpretation I' can be induced from I where $I', 0 \models \alpha$, $size(I') \leq |\alpha|$, $final(I') = 1$ and the preferential relation $\prec'$ is only on time points within the finite sequence. As discussed after Definition 11.2, a finite preferential structure S can be induced from I'. Therefore, we can make a non-deterministic guess for a finite preferential structure $S = (n, V_S, \prec_S)$ s.t. $size(S) \leq |\alpha|$. Next, for each $\alpha_1 \in Sf(\alpha)$ in the increasing order of $|\alpha_1|$ and for each $t \in [0, n-1]$, we update $lab^S_\alpha(t)$. At the end of this procedure, S is accepted as a model for α iff $\alpha \in lab^S_\alpha(0)$, otherwise, S is rejected (thanks to Corollary 11.6).

The procedure is polynomial-time bounded. Since the set $Sf(\alpha)$ is ordered by the increasing length of sub-sentences of α, then each time we want to add a sub-sentence α_1 to $lab^S_\alpha(t)$, the presence of all of the sub-sentences of α_1 in the labelling set $lab^S_\alpha(t)$ has already been checked for all $t \in [0, n-1]$. Therefore, checking whether said sub-sentences of α_1 are in a point t' is a simple "yes" or "no" question. Thus, for each sub-sentence α_1 and $t \in [0, n-1]$, we check only once if $\alpha_1 \in lab^S_\alpha(t)$. We can see that the most costly sentence to check time wise is $\diamondsuit$ sentences. Say that we check for a sentence $\diamondsuit\alpha_1 \in lab^S_\alpha(t)$ with $t \in [0, n-1]$. In this case, we need to check whether there is a $t' \in [t, n-1]$ s.t. $\alpha_1 \in lab^S_\alpha(t')$ (which costs $\mathcal{O}((t-n))$) and $t' \in \min_{\prec^S}(t)$ (which costs $\mathcal{O}((t-n))$). In the worst case scenario, this takes a time of $\mathcal{O}((t-n)^2)$. Since $size(S) = n$, checking whether $\diamondsuit\alpha_1 \in lab^S_\alpha(0)$ costs at most $\mathcal{O}(n^2)$, checking whether $\diamondsuit\alpha_1 \in lab^S_\alpha(1)$ costs at most $\mathcal{O}((n-1)^2)$, and so on. If we add them together, then for all $t \in [0, n-1]$, checking whether $\diamondsuit\alpha_1 \in lab^S_\alpha(t)$ is $\mathcal{O}(n^3)$. Checking whether $\Box\alpha_1$ and $\Diamond\alpha_1$ sentences for all $t \in [0, n-1]$ costs at most $\mathcal{O}(n^2)$. We only check for the presence of the sub-sentence α_1 in $lab^S_\alpha(t')$ with $t' \in [t, n-1]$. For $\bigcirc$ and Boolean sentences, it costs at most $\mathcal{O}(n)$. Suppose that $|Sf(\alpha)| = k$, checking for all sub-sentences of α for all $t \in [0, n-1]$ costs $\mathcal{O}(k * n^3)$. Without loss of generality, since $k, n \leq |\alpha|$, then the full expansion of the labelling sets costs $\mathcal{O}(|\alpha|^4)$ at most.

Theorem 11.7. *$\mathfrak{I}$-satisfiability for $\mathcal{L}_1$ sentences is NP-complete.*

Proof. $\mathfrak{I}$-satisfiability for $\mathcal{L}_1$ sentences is at least NP-hard because the satisfiability of Boolean sentences is an NP-hard problem, and Boolean sentences are a subset of $\mathcal{L}_1$.

Let $\alpha \in \mathcal{L}_1$. If α is $\mathfrak{I}$-satisfiable, then there is an interpretation $I = (V, \prec) \in \mathfrak{I}$ s.t. $I, 0 \models \alpha$. Thanks to Theorem 10.5, an interpretation $I' = (V', \prec)$ where $size(I') \leq |\alpha|$ and $I', 0 \models \alpha$ can be induced from I . The interpretation I' can be represented by a finite preferential structure $S = (n, V_S, \prec_S)$ where $\mathsf{I}(S)$ is I'. We make a non-deterministic guess of a finite preferential structure $S = (n, V_S, \prec_S)$ where $size(S) \leq |\alpha|$ and use the labelling sets $lab^S_\alpha(t)$ to check for all sub-sentences of α_1 in each $t \in [0, n-1]$. Thanks to Corollary 11.6, if $\alpha \in lab^S_\alpha(0)$, then S is accepted as a model and therefore α is satisfiable. Otherwise, S is rejected. Using the aforementioned procedure, the labelling sets is polynomial-time bounded in $\mathcal{O}(|\alpha|^4)$. $\mathfrak{I}$-satisfiability for $\mathcal{L}_1$ sentences is an NP problem. Therefore, $\mathfrak{I}$-satisfiability for $\mathcal{L}_1$ sentences is NP-complete. $\square$

12. The bounded-model property of the fragment $\mathcal{L}_2$

The second contribution of our work is to show the decidability of the satisfiability problem of another fragment of defeasible LTL, namely $\mathcal{L}_2$ (See Section 7.2). Just as a reminder, sentences in $\mathcal{L}_2$ are recursively defined as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \alpha \vee \alpha \mid \Box\alpha \mid \Diamond\alpha \mid \boxtimes\alpha \mid \Diamond\!\!\!\!\sim\alpha$$

Let $\alpha \in \mathcal{L}_2$ be a sentence. With $|\alpha|$ we denote the number of symbols within α. The main result of this section is summarized in the following theorem, of whose proof will be given in the remainder of the section.

Theorem 12.1 (Bounded-model property). *If $\alpha \in \mathcal{L}_2$ is $\mathfrak{I}^{sd}$-satisfiable, then there is an interpretation $I \in \mathfrak{I}^{sd}$ such that $I, 0 \models \alpha$ and $size(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$.*

Hence, given a $\mathfrak{I}^{sd}$-satisfiable sentence $\alpha \in \mathcal{L}_2$, there is an $\mathfrak{I}^{sd}$-interpretation satisfying α whose size is bounded. Since α is $\mathfrak{I}^{sd}$-satisfiable, we know $I, 0 \models \alpha$. From I we can construct an interpretation I' also satisfying α, i.e., $I', 0 \models \alpha$, which is bounded on its size by $|\alpha| \times 2^{|\mathcal{P}|}$.

The goal of this section is to show how to build said bounded interpretation. Let $\alpha \in \mathcal{L}_2$ and let $I \in \mathfrak{I}^{sd}$ be s.t. $I, 0 \models \alpha$. The first step is to characterize an acceptable sequence N w.r.t. I such that N is bounded first of all, and "keeps" the satisfiability of the sub-sentences α_1 contained in α, i.e., if $I, t \models \alpha_1$, then $I^N, t \models_{\mathscr{P}} \alpha_1$ (see Definition 9.2). We do so by building inductively a bounded pseudo-interpretation step by step by selecting what to take from the initial interpretation I for each sub-sentence α_1 contained in α to be satisfied. In what follows, we introduce the notion of $Anchors(\cdot)$ as a strategy for picking out the desired time points from I. Definitions 12.4–12.8 tell us how to pick these time points.

Definition 12.2 (Induced acceptable sequence). Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let N be a sequence of time points. Let N' be the sequence of all time points t' in *final*(I) for which there is $t \in N \cap \mathit{final}(I)$ with $V(t') = V(t)$. With $AS(I, N) \stackrel{\text{def}}{=} N \cup N'$ we denote the *induced acceptable sequence* of N w.r.t. I.

Example 12.3. Let $I = (V, \prec)$ be the interpretation represented in Figure 4 and N be a sequence such that $N = (t_0, t_1, t_2)$ (marked with black circles on the figure). In order to obtain $AS(I, N)$, we look for time points of the sequence N that are in *final*(I).

The only time point in $\mathit{final}(I)$ is t_2 and has the valuation $V(t_2) = V_1$. In addition of t_0, t_1, t_2, the induced acceptable sequence of N w.r.t. I, denoted by $AS(I, N)$, contains all time points in $\mathit{final}(I)$ that have the same valuation as t_1 (marked with green circles on the figure).

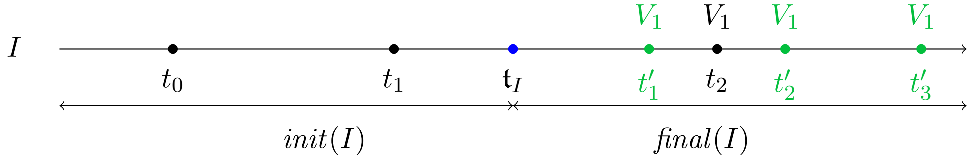


Figure 4.: Induced acceptable sequence

In the previous definition, N' is the sequence of all time points t' having the same valuation as some time point $t \in N$ that is in $\mathit{final}(I)$. It is also worth to point out that N' can be empty in the case of there being no time point $t \in N$ that is in $\mathit{final}(I)$. N is then a finite acceptable sequence w.r.t. I where $AS(I, N) = N$. This notation is mainly used to ensure that we are using the acceptable version of any sequence.

Definition 12.4 (Chosen occurrence w.r.t. α)**.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $\alpha \in \mathcal{L}_2$ and N be an acceptable sequence w.r.t. I s.t. there exists a time point t in N with $I, t \models \alpha$. The *chosen occurrence* satisfying α in N, denoted by $\mathfrak{t}_{\alpha}^{I,N}$, is defined as follows:

$$\mathfrak{t}_{\alpha}^{I,N} \stackrel{\text{def}}{=} \begin{cases} \min_{<}\{t \in \mathit{final}(I, N) \mid I, t \models \alpha\}, & \text{if } \{t \in \mathit{final}(I, N) \mid I, t \models \alpha\} \neq \emptyset; \\ \max_{<}\{t \in \mathit{init}(I, N) \mid I, t \models \alpha\}, & \text{otherwise.} \end{cases}$$

Notice that $<$ above denotes the natural ordering of the underlying temporal structure. The strategy to pick out a time point satisfying a given sentence α in N is as follows. If such sentence is in the final part, we pick the first time point that satisfies it, since we have the guarantee to find infinitely many time points having the same valuation as $\mathfrak{t}_{\alpha}^{I,N}$ that also satisfy α (see Lemma 9.7). If not, we pick the last occurrence in the initial part that satisfies α. Thanks to Definition 12.4, we can limit the number of time points taken that satisfy the same sentence.

Example 12.5. To highlight the notion of chosen occurrence, we illustrate it in Figure 5. On the figure, the coloured circles points are the time points of the sequence N that satisfy α_1.

In Case 1, both t and t' are in $\mathit{init}(I)$. We pick the last occurrence which is t' (coloured in blue) as the chosen occurrence $\mathfrak{t}_{\alpha_1}^{I,N} = t'$.

In Case 2, all of the time points of N that satisfy α are in $\mathit{final}(I)$. We pick the first occurrence in $N \cap \mathit{final}(I)$, which is t'_1 as the chosen occurrence $\mathfrak{t}_{\alpha_1}^{I,N}$.

In Case 3, even when time points of N are both in $\mathit{init}(I)$ and $\mathit{final}(I)$, the chosen occurrence $\mathfrak{t}_{\alpha_1}^{I,N}$ is the first time point in $N \cap \mathit{final}(I)$ that satisfies α_1.

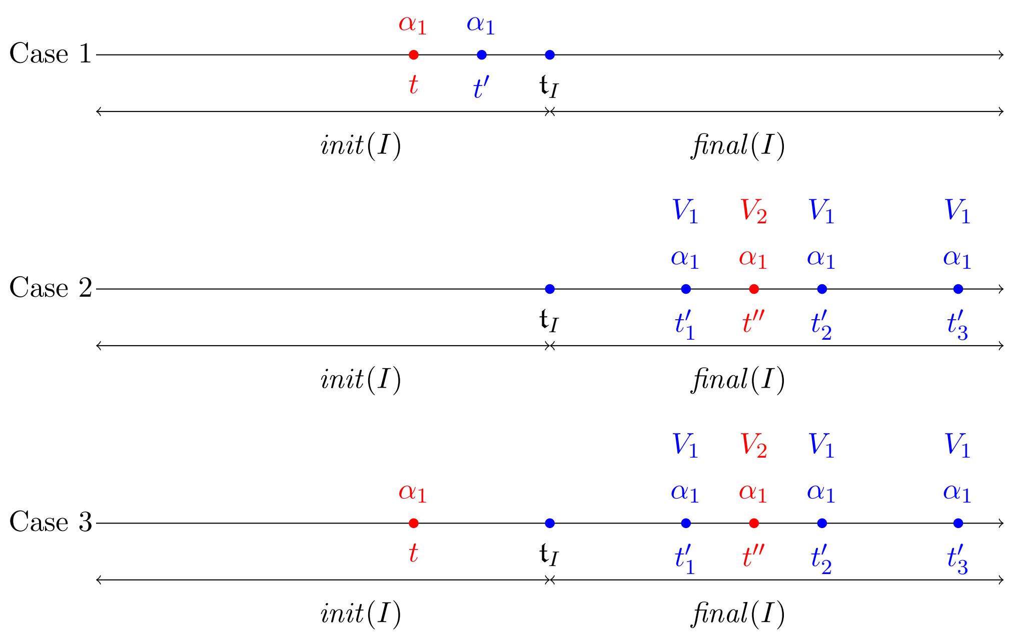


Figure 5.: Selected time points of α in $AS(I, N)$

Next, we define the sequence $ST(\cdot)$ as the induced acceptable sequence of the sequence that contains only the chosen occurrence.

Definition 12.6 (Selected time points)**.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, N be an acceptable sequence w.r.t. I and $\alpha \in \mathcal{L}_2$ s.t. there is t in N s.t. $I, t \models \alpha$. With $ST(I, N, \alpha) \stackrel{\text{def}}{=} AS(I, (\mathsf{t}_\alpha^{I,N}))$ we denote the *selected time points* of N and α w.r.t. I. (Note that $(\mathsf{t}_\alpha^{I,N})$ is a sequence of only one element.)

Example 12.7. In Example 12.5, we obtained the chosen occurrences for each of the cases represented in Figure 5. The next step is to compute $ST(I, N, \alpha_1)$.

In Case 1, since $ST(I, N, \alpha_1)$ is the induced acceptable sequence of $(\mathsf{t}_{\alpha_1}^{I,N})$ and $\mathsf{t}_{\alpha_1}^{I,N} \in init(I)$, then $ST(I, N, \alpha_1) = (\mathsf{t}_{\alpha_1}^{I,N}) = (t')$.

In Case 2, now that $\mathsf{t}_{\alpha_1}^{I,N}$ is in $final(I)$, the sequence $ST(I, N, \alpha_1)$ is the acceptable sequence w.r.t. I that contains all time points $final(I)$ with the same valuation as $\mathsf{t}_{\alpha_1}^{I,N}$ (coloured in blue in Figure 5), i.e., $ST(I, N, \alpha_1) = (t'_1, t'_2, t'_3, \cdots)$ with $V(t'_i) = V_1$ for all $i \geq 1$.

In Case 3 and following the same line of reasoning as in Case 2, since we have $\mathsf{t}_{\alpha_1}^{I,N} = t'_1$ and $\mathsf{t}_{\alpha_1}^{I,N} \in final(I)$, then $ST(I, N, \alpha_1)$ is the sequence $(t'_1, t'_2, t'_3, \cdots)$ with $V(t'_i) = V_1$ for all $i \geq 1$ (coloured in blue in Figure 5).

Given a sentence $\alpha \in \mathcal{L}_2$ and an acceptable sequence N w.r.t. I s.t. there is at least one time point $t \in N$ where $I, t \models \alpha$, the sequence $ST(I, N, \alpha)$ is the induced acceptable sequence of the sequence $(\mathsf{t}_\alpha^{I,N})$. If $\mathsf{t}_\alpha^{I,N} \in init(I)$, the sequence $ST(I, N, \alpha)$ is the sequence $(\mathsf{t}_\alpha^{I,N})$. Otherwise, the sequence $ST(I, N, \alpha)$ is the sequence of all time points t in $final(I)$ that have the same valuation as $\mathsf{t}_\alpha^{I,N}$. In both cases, we can see that $size(I, ST(I, N, \alpha)) = 1$.

Given an interpretation $I = (V, \prec)$ and N an acceptable sequence w.r.t I, the *representative sentence* of a valuation v is formally defined as $\alpha_v \stackrel{\text{def}}{=} \bigwedge\{p \mid p \in v\} \wedge$

$\bigwedge\{\neg p \mid p \notin v\}$.

Definition 12.8 (Distinctive reduction)**.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let N be an acceptable sequence w.r.t. I. With $DR(I,N) \stackrel{\text{def}}{=} \bigcup_{v \in val(I,N)} ST(I,N,\alpha_v)$ (The definition of $val(I,N)$ can be found in Definition 9.11) we denote the distinctive reduction of N.

Given an acceptable sequence N w.r.t. I, $DR(I,N)$ is the sequence containing the chosen occurrence $\mathfrak{t}^{I,N}_{\alpha_v}$ that satisfies the representative α_v in N for each $v \in val(I,N)$. In other words, we pick the selected time points for each possible valuation in $val(I,N)$. There are two interesting results with regard to $DR(I,N)$. The first one is that $DR(I,N)$ is an acceptable sequence w.r.t. I. This can easily be proven since $ST(I,N,\alpha_v)$ is also an acceptable sequence w.r.t. I, and the union of all $ST(I,N,\alpha_v)$ is an acceptable sequence w.r.t. I (see Proposition 9.12). The second result is that $size(I,DR(I,N)) \leq 2^{|\mathcal{P}|}$. Indeed, thanks to Proposition 9.12, we can see that $size(I,DR(I,N)) \leq \sum_{v \in val(I,N)} size(I,ST(I,N,\alpha_v))$. Moreover, we have $size(I,ST(I,N,\alpha_v)) = 1$ for each $v \in val(I,N)$. On the other hand, there are at most $2^{|\mathcal{P}|}$ possible valuations in $val(I,N)$. Thus, we can assert that $\sum_{v \in val(I,N)} size(I,ST(I,N,\alpha_v)) \leq 2^{|\mathcal{P}|}$, and then we have $size(I,DR(I,N)) \leq 2^{|\mathcal{P}|}$.

Definition 12.9 (Anchors)**.** Let $\alpha \in \mathcal{L}_2$ be of the form $\mathcal{O}\alpha_1$ where $\mathcal{O} \in \{\Diamond, \Box, \tilde{\Diamond}, \tilde{\Box}\}$ and $\alpha_1 \in \mathcal{L}_2$. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. The sequence $Anchors(I,T,\alpha)$ is defined as:

$$\begin{aligned}
Anchors(I,T,\Diamond\alpha_1) &\stackrel{\text{def}}{=} ST(I,\mathbb{N},\alpha_1);\\
Anchors(I,T,\Box\alpha_1) &\stackrel{\text{def}}{=} \emptyset;\\
Anchors(I,T,\tilde{\Diamond}\alpha_1) &\stackrel{\text{def}}{=} \textstyle\bigcup_{t \in T} ST(I,AS(I,\min_{\prec}(t)),\alpha_1);\\
Anchors(I,T,\tilde{\Box}\alpha_1) &\stackrel{\text{def}}{=} DR(I,\textstyle\bigcup_{t \in T} AS(I,\min_{\prec}(t))).
\end{aligned}$$

Given an acceptable sequence T w.r.t. $I \in \mathfrak{I}^{sd}$ where all of its time points satisfy $\mathcal{O}\alpha_1$ (where $\mathcal{O} \in \{\Diamond, \Box, \tilde{\Diamond}, \tilde{\Box}\}$), $Anchors(I,T,\mathcal{O}\alpha_1)$ is an acceptable sequence w.r.t. I such that all of its elements have the sub-sentence α_1. The goal here if we inductively select the time points that satisfy α_1 in $Anchors(I,T,\mathcal{O}\alpha_1)$, all of the $\mathcal{O}\alpha_1$ sentences in T would then be satisfied. We shall start with $Anchors(\cdot)$ for $\Diamond\alpha_1$ sentences (see Figure 6). Let T be an acceptable sequence w.r.t. I such that all of its elements have the sentence $\Diamond\alpha_1$. In case 1 of Figure 6, let N be an acceptable sequence that contains t_0, t_1, t_2 and t' such that $I^N, t' \models_{\mathscr{P}} \alpha_1$. The sentence $\Diamond\alpha_1$ is then satisfied in I^N ,i.e., $I^N, t_0 \models_{\mathscr{P}} \Diamond\alpha_1$ (same goes for t_1, t_2). We can see that t'' is also a candidate that keeps the satisfiability of $\Diamond\alpha_1$ in t_0, t_1 and t_2. However, in order to have a bounded number of elements, we use the selected time points $ST(\cdot)$ function. If all time points that satisfy α_1 are in the *init*(I), we pick the last one (t' in case 1). Otherwise, we choose the first candidate that satisfies α_1 in *final*(I) (t'_1 in case 2) and pick all time points of *final*(I) with the same valuation as t'_1. Even if such candidate comes before t_i with $\Diamond\alpha_1$ (case 3), there is always a time point with the same valuation that comes after t_i that satisfies α_1 (Thanks to Lemma 9.6, time points with the same valuation in *final*(I) satisfy the same sentences). By choosing the first when it comes to *final*(I) and the last when it comes to *init*(I), the picked can overlap with each other. This is the essence of $ST(\cdot)$ function and our strategy for picking time points.

It is worth to point out that the choice of $Anchors(I,T,\Box\alpha_1) = \emptyset$ is due to the fact α_1 is satisfied starting from the first time point t_0 in T and onwards, i.e., for all $t \geq t_0$,

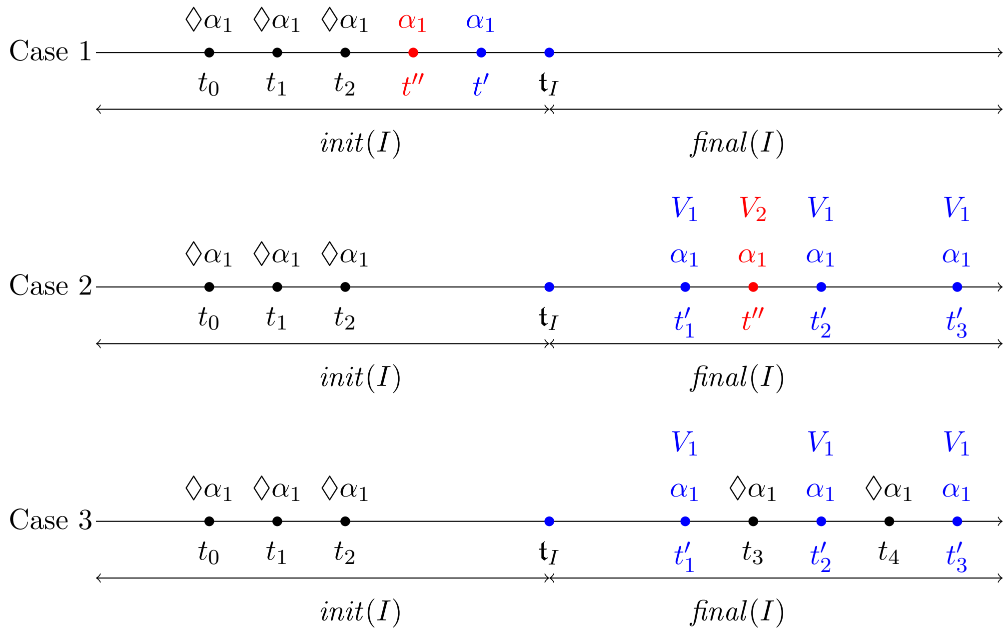


Figure 6.: *Anchors* for $\Diamond$-sentences

we have $I, t \models \alpha_1$. We need to make sure that the sentence $\Box\alpha_1$ remains satisfied in $t_i \in T$ for all pseudo-interpretations I^N where $T \subseteq N$.

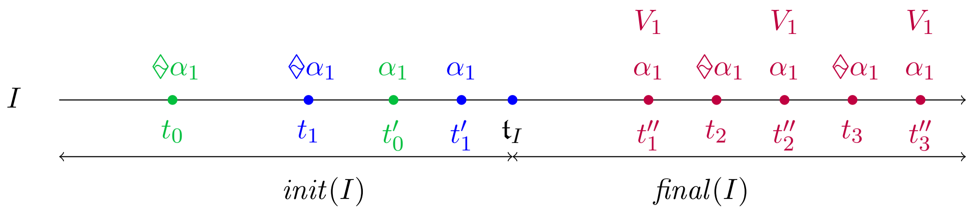


Figure 7.: *Anchors* for $\diamondminus$-sentences

Moving on to $Anchors(\cdot)$ for $\diamondminus\alpha_1$ sentences (see Figure 7), each time point of t_i in $init(I, T)$ is represented by different color in the figure. Time points $t_j \in final(I, T)$ have the same color. For each time point $t_i \in T$, we shall pick the selected time points (using the $ST(\cdot)$ function) in $\min_{\prec}(t_i)$ that satisfy α_1. In Figure 7, each time point t_i and its selected candidate have the same color. Similarly to $\Diamond\alpha_1$, if there is an acceptable sequence N that satisfies α_1 in all of the chosen occurrences. Then, the sentence $\diamondminus\alpha_1$ is also satisfied. Note that all the picked candidates need to be minimal to their respective t_i w.r.t. $\prec$. Later on this section, we will show $Anchors(\cdot)$ for $\diamondminus\alpha_1$ sentences is an acceptable sequence w.r.t. I that is bounded in its size.

Finally, $Anchors(\cdot)$ for $\boxminus\alpha_1$ sentences is represented in Figure 8. For each time t_i (for simplicity's sake, there is only one time point t_0 in Figure 8), the selected time points are the chosen occurrence for each distinct valuation in $\min_{\prec}(t_i)$, i.e., V_1, V_2, V_3, V_4 (each distinct valuation is represented by a different color in Figure 8). Note that the all the selected time points have the sentence α_1. These candidates have a particular property that we shall motivate in Proposition 12.12.

The following are some properties of $Anchors(\cdot)$ sequence:

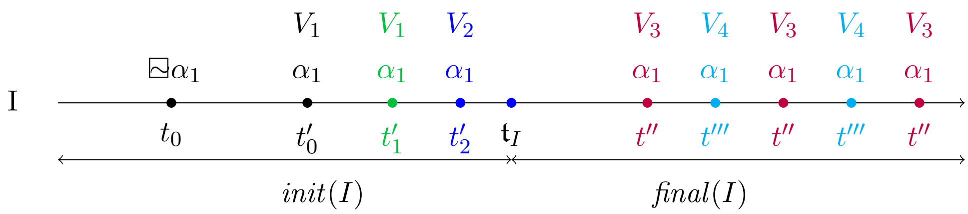


Figure 8.: *Anchors* for $\Box\!\!\!\!\sim$-sentences

Lemma 12.10. *Let $\alpha_1 \in \mathcal{L}_2$ be a sentence, $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let T be a non-empty acceptable sequence w.r.t. I where for all $t_i \in T$ we have $I, t_i \models \Diamond\!\!\!\!\sim\alpha_1$. Then for all $t, t' \in$ Anchors$(I, T, \Diamond\!\!\!\!\sim\alpha_1)$ s.t. $V(t) = V(t')$ and $t \neq t'$, we have $t, t' \in$ final$(I,$ Anchors$(I, T, \Diamond\!\!\!\!\sim\alpha_1))$.*

Proposition 12.11. *Let $\alpha \in \mathcal{L}_2$ be of the form $\mathcal{O}\alpha_1$, where $\mathcal{O} \in \{\Diamond, \Box, \Diamond\!\!\!\!\sim, \Box\!\!\!\!\sim\}$ and $\alpha_1 \in \mathcal{L}_2$. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models \alpha$. Then, we have:*

$$\mathit{size}(I, \mathit{Anchors}(I, T, \alpha)) \leq 2^{|\mathcal{P}|}.$$

Proposition 12.12. *Let $\alpha_1 \in \mathcal{L}_2$, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \Box\!\!\!\!\sim\alpha_1$, with $\alpha_1 \in \mathcal{L}_2$. For all acceptable sequences N w.r.t. I s.t. Anchors$(I, T, \Box\!\!\!\!\sim\alpha_1) \subseteq N$ and for all $t_i \in N \cap T$, let $I^N = (V^N, \prec^N)$ be the pseudo-interpretation over N and $t' \in N$. We have the following:*

$$\text{If } t' \notin \min_{\prec}(t_i), \text{ then } t' \notin \min_{\prec^N}(t_i).$$

When trying to build the bounded pseudo-interpretation I^N that satisfies α, one problem we encountered is that the set $\min_{\prec^N}(t)$ may include a time point t' that is not in $\min_{\prec}(t)$. This becomes an issue when checking truth values of defeasible sentences in I^N. In order to solve this issue, we defined *Anchors*$(\cdot)$ to pick only time points such that we keep truth values of defeasible sentences. In the case of $\Diamond\!\!\!\!\sim$-sentences, the sequence *Anchors*$(I, T, \Diamond\!\!\!\!\sim\alpha_1)$ contains the selected time point t'_i that satisfies α_1 and is minimal to t_i w.r.t. $\prec$ for each $t_i \in T$. This is sufficient to preserve the truth $\Diamond\!\!\!\!\sim\alpha_1$ for each $t_i \in T$. As for $\Box\!\!\!\!\sim$-sentences and for each $t_i \in T$, the sequence *Anchors*$(I, T, \Box\!\!\!\!\sim\alpha_1)$ contains selected time points t'_i for each distinct valuation in $\min_{\prec}(t_i)$. As showed in Proposition 12.12, any time point t'_i that is not originally in $\min_{\prec}(t_i)$, is therefore not in $\min_{\prec^N}(t_i)$.

With *Anchors*$(\cdot)$ defined, we introduce the notion of *Keep*$(\cdot)$. The sequence *Keep*$(\cdot)$ will help us to compute recursively, starting from the initial satisfiable sentence α down to its literals, the selected time points to pick in order to induce the pseudo-interpretation I^N that is bounded in size and satisfies α.

Definition 12.13 (Keep)**.** Let $\alpha \in \mathcal{L}_2$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be an acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. The sequence *Keep*(I, T, α) is defined as $\emptyset$, if $T = \emptyset$; otherwise it is recursively defined as follows:

- $\mathit{Keep}(I, T, \ell) \stackrel{\text{def}}{=} \emptyset$, where ℓ is a literal;
- $\mathit{Keep}(I, T, \alpha_1 \wedge \alpha_2) \stackrel{\text{def}}{=} \mathit{Keep}(I, T, \alpha_1) \cup \mathit{Keep}(I, T, \alpha_2)$;

- $\mathit{Keep}(I,T,\alpha_1 \vee \alpha_2) \stackrel{\text{def}}{=} \mathit{Keep}(I,T_1,\alpha_1) \cup \mathit{Keep}(I,T_2,\alpha_2)$, where $T_1 \subseteq T$ (resp. $T_2 \subseteq T$) is the sequence of all $t_1 \in T$ (resp. $t_2 \in T$) s.t. $I,t_1 \models \alpha_1$ (resp. $I,t_2 \models \alpha_2$);
- $\mathit{Keep}(I,T,\Diamond\alpha_1) \stackrel{\text{def}}{=} \mathit{Anchors}(I,T,\Diamond\alpha_1) \cup \mathit{Keep}(I,\mathit{Anchors}(I,T,\Diamond\alpha_1),\alpha_1)$;
- $\mathit{Keep}(I,T,\Box\alpha_1) \stackrel{\text{def}}{=} \mathit{Keep}(I,T,\alpha_1)$;
- $\mathit{Keep}(I,T,\,\Diamond\!\!\!\!\cdot\;\alpha_1) \stackrel{\text{def}}{=} \mathit{Anchors}(I,T,\,\Diamond\!\!\!\!\cdot\;\alpha_1) \cup \mathit{Keep}(I,\mathit{Anchors}(I,T,\,\Diamond\!\!\!\!\cdot\;\alpha_1),\alpha_1)$;
- $\mathit{Keep}(I,T,\,\boxdot\alpha_1) \stackrel{\text{def}}{=} \mathit{Anchors}(I,T,\,\boxdot\alpha_1) \cup \mathit{Keep}(I,T',\alpha_1)$, where $T' = \bigcup_{t_i \in T} AS(I,\min_{\prec}(t_i))$.

With $\mu(\alpha)$ we denote the number of classical and non-monotonic modalities in α.

Proposition 12.14. *Let $\alpha \in \mathcal{L}_2$ be in NNF, $I = (V,\prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I,t \models \alpha$. Then, we have $\mathit{size}(I,\mathit{Keep}(I,T,\alpha)) \leq \mu(\alpha) \times 2^{|\mathcal{P}|}$.*

Given an acceptable sequence N w.r.t. I, we need to make sure that for each added time point t in the induced pseudo-interpretation I^N, we keep the truth values of the sub-sentences in t, i.e., if $I,t \models \alpha$, then $I^N,t \models_{\mathscr{P}} \alpha$. The sequence $\mathit{Keep}(I,T,\alpha)$ is the acceptable sequence of time points s.t. if $\mathit{Keep}(I,T,\alpha) \subseteq N$ and $t \in T$, then said condition is met. We prove this in Lemma 12.15.

Lemma 12.15. *Let $\alpha \in \mathcal{L}_2$ be in NNF, $I = (V,\prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I,t \models \alpha$. For all acceptable sequences N w.r.t. I, if $\mathit{Keep}(I,T,\alpha) \subseteq N$, then for every $t \in N \cap T$, we have $I^N,t \models_{\mathscr{P}} \alpha$.*

Since we build our pseudo-interpretation I^N by adding selected time points for each sub-sentence α_1 of α, we need to make sure that such sub-sentence remains satisfied in I^N. Lemma 12.15 ensures that.

Definition 12.16 (Induced interpretation)**.** Let $I = (V,\prec) \in \mathfrak{I}^{sd}$ and let N be an infinite acceptable sequence w.r.t. I and $t_i,t_j \in N$. The interpretation $I' = (V',\prec') \in \mathfrak{I}^{sd}$ is induced from the pseudo-interpretation $I^N = (V^N,\prec^N)$ as follows:

- for all $i \geq 0$, we have $V'(i) = V^N(t_i)$;
- for all $i,j \geq 0$, $t_i,t_j \in N$, we have $(t_i,t_j) \in \prec^N$ iff $(i,j) \in \prec'$.

It is worth mentioning that the I^N-induced interpretation I' is a state-dependent interpretation. Moreover, we have $\mathit{size}(I') = \mathit{size}(I^N)$. We notice also that if $I^N,t_0 \models \alpha$, then $I',0 \models \alpha$. We can now prove our bounded-model theorem.

Proof of Theorem 12.1. We assume that $\alpha \in \mathcal{L}_2$ is $\mathfrak{I}^{sd}$-satisfiable. The first thing we notice is that $|\alpha| \geq \mu(\alpha)+1$. Let α' be the NNF of the sentence α. As a consequence of the duality rules of $\mathcal{L}_2$, we can deduce that $\mu(\alpha') = \mu(\alpha)$. Let $I = (V,\prec) \in \mathfrak{I}^{sd}$ s.t. $I,0 \models \alpha'$. Let $T_0 = AS(I,(0))$ be an acceptable sequence w.r.t. I. We can see that $\mathit{size}(I,T_0) = 1$. Since for all $t \in T_0$ we have $I,t \models \alpha'$ (see Lemma 9.7), we can compute recursively $U = \mathit{Keep}(I,T_0,\alpha')$. Thanks to Proposition 12.14, we conclude that U is an acceptable sequence w.r.t. I s.t. $\mathit{size}(I,U) \leq \mu(\alpha') \times 2^{|\mathcal{P}|}$. Let $N = T_0 \cup U$ be the union of T_0 and U and let $I^N = (V^N,\prec^N)$ be its pseudo-interpretation over N. Thanks to Proposition 9.12, we have $\mathit{size}(I,N) \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Thanks to Lemma 12.15, since $0 \in N \cap T_0$ and $\mathit{Keep}(I,T_0,\alpha') \subseteq N$, we have $I^N,0 \models_{\mathscr{P}} \alpha'$. In case N is finite, we replicate the last time point t_n infinitely many times. Notice that $\mathit{size}(I,N)$ does not change if we replicate the last element. We obtain the I^N-induced interpretation $I' \in \mathfrak{I}^{sd}$ by changing the labels of N into a sequence of natural numbers minding the order of time points in N (see Definition 12.16). We can see that $size(I') = size(I, N)$ and $I', 0 \models \alpha$. Consequently, we have $size(I') \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Hence, from a given interpretation I s.t. $I, 0 \models \alpha$ we can build an interpretation I' s.t. $I', 0 \models \alpha$ and $size(I') \leq 1 + \mu(\alpha') \times 2^{|\mathcal{P}|}$. Since $|\alpha| \geq \mu(|\alpha|) + 1$ and $\mu(|\alpha'|) = \mu(|\alpha|)$, we conclude that $size(I') \leq |\alpha| \times 2^{|\mathcal{P}|}$. □

13. The satisfiability problem in $\mathcal{L}_2$

We discuss in this section the satisfiability checking problem of the fragment $\mathcal{L}_2$. We use a similar procedure as described in Section 11. Given a sentence α, a bounded structure is non-deterministically guessed, then the labeling sets $lab^S_\alpha(\cdot)$ are used to update the set of sub-sentences in each element of the sequence and checking whether the sentence α is satisfied. Thanks to Theorem 12.1, if a sentence $\alpha \in \mathcal{L}_2$ is $\mathfrak{I}^{sd}$-satisfiable, then there exists an interpretation $I \in \mathfrak{I}^{sd}$ s.t. $size(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$ that satisfies the sentence. We use a compact structure to represent state-dependent interpretations. For this purpose, we focus on particular interpretations of the class $\mathfrak{I}^{sd}$, namely the ultimately periodic interpretations (UPI in short). We show that any interpretation $I \in \mathfrak{I}^{sd}$ has an equivalent UPI. As we will see in the second part of this section, we define a finite representation of UPIs, called finite preferential structures.

Definition 13.1 (UPI)**.** Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $\pi = card(range(I))$. We say I is an *ultimately periodic interpretation* if:

- for every $t, t' \in [\mathsf{t}_I, \mathsf{t}_I + \pi[$ s.t. $t \neq t'$, we have $V(t) \neq V(t')$,
- for every $t \in [\mathsf{t}_I, \infty[$, we have $V(t) = V(\mathsf{t}_I + (t - \mathsf{t}_I) \bmod \pi)$.

A UPI I is a state-dependent interpretation s.t. each time point's valuation in $final(I)$ is replicated periodically. Given a UPI, $\pi = card(range(I))$ denotes the length of the period and the interval $[\mathsf{t}_I, \mathsf{t}_I + \pi[$ is the first period which is replicated periodically throughout the final part. It is worth pointing out that for every $t \in final(I)$, we have $V(t) \in \{V(t') \mid t' \in [\mathsf{t}_I, \mathsf{t}_I + \pi[\}$, which is one of the consequences of the definition above. Thanks to Lemma 9.9, we can prove the following proposition.

Proposition 13.2. *Let $\mathcal{P}$ be a set of atomic propositions, $I = (V, \prec) \in \mathfrak{I}^{sd}$, $i = length(init(I))$ and $\pi = card(range(I))$. There exists an ultimately periodic interpretation $I' = (V', \prec') \in \mathfrak{I}^{sd}$ s.t. I, I' are faithful interpretations over $\mathcal{P}$ (see Definition 9.8), $init(I') \doteq init(I)$, $range(I') = range(I)$ (see Definition 9.8 and Lemma 9.9 for reference) and $V'(0) = V(0)$. Moreover, for all $\alpha \in \mathcal{L}_2$, we have $I, 0 \models \alpha$ iff $I', 0 \models \alpha$.*

It is worth to point out that the size of an interpretation and that of its UPI are the same. It can easily be seen that these interpretations have the same initial part and the same range of valuations in the final part. I' from the aforementioned proposition is obtained from I by keeping the same initial part, and placing each distinct valuation of $range(I)$ in the interval $[\mathsf{t}_I, \mathsf{t}_I + \pi[$ and finally replicating this interval infinitely many times. Moreover, the preference relation $\prec'$ arranges valuations in the same way as $\prec$. We can see that I and I' are faithful and that $init(I') \doteq init(I)$, $range(I') = range(I)$ and $V'(0) = V(0)$. Therefore, I and its UPI I' satisfy the same sentences.

We showed that, starting from any interpretation $I \in \mathfrak{I}^{sd}$, the equivalent UPI can be induced. Next, we introduce a compact representation for ultimately periodic structures. The structure used for checking the satisfiability of $\mathcal{L}_2$ sentences is defined in the following way:

Definition 13.3 (Finite preferential structure)**.** A finite preferential structure is a tuple $S = (i, \pi, V_S, \prec_S)$ where: i, π are two integers such that $i \geq 0$ and $\pi > 0$ (where i is intended to be the starting point of the period, π is the length of the period); $V_S : [0, i+\pi[\longrightarrow 2^{\mathcal{P}}$, and $\prec_S \;\subseteq 2^{\mathcal{P}} \times 2^{\mathcal{P}}$ is a strict partial order. Moreover, (I) for all $t \in [i, i+\pi[$, we have $V_S(t) \neq V_S(i-1)$; and (II) for all distinct $t, t' \in [i, i+\pi[$, we have $V_S(t) \neq V_S(t')$.

The structure is split into two intervals. The interval $[0, i[$ represents the initial part of a UPI I, and the interval $[i, i+\pi[$ is the first period of the final part of I. Each element in the interval $[i, i+\pi[$ has a unique valuation, they represent all valuations in the range of I. We suppose that the elements of $[i, i+\pi[$ are inter-connected. Since this interval is infinitely replicated in the final part of the interpretation, then every time point with a valuation in $[i, i+\pi[$ is a successor of all time points with valuations in $[i, i+\pi[$. The added properties (I) and (II) make sure that we can build a structure S from a UPI I, and back (the initial part of I coincides with $[0, i[$ and the valuations in $[i, i+\pi[$ are the range of final part of I). Starting from a structure S, we can build a UPI $\mathsf{I}(S)$ as follows:

Definition 13.4. Given a finite preferential structure $S = (i, \pi, V_S, \prec_S)$, let $\mathsf{I}(S) \stackrel{\text{def}}{=} (V, \prec)$, where for every $t \geq 0$, $V(t) \stackrel{\text{def}}{=} V_S(t)$, if $t < i$, and $V(t) \stackrel{\text{def}}{=} V_S(i + (t-i) \bmod \pi)$, otherwise. The ordering relation $\prec$ is defined as $\prec \stackrel{\text{def}}{=} \{(t, t') \mid (V(t), V(t')) \in \prec_S\}$.

Given a structure $S = (i, \pi, V_S, \prec_S)$, the interval $[0, i[$ of the structure corresponds to the initial temporal part of the underlying interpretation $\mathsf{I}(S)$ and $[i, i+\pi[$ represents a temporal period that is infinitely replicated in order to determine the final temporal part of the interpretation. The ordering relation $\prec$ of $\mathsf{I}(S)$ is the projection of $\prec_S$ over the time points in the sequence. It follows directly from $\prec_S$ being an ordering relation on valuations that the relation $\prec$ of $\mathsf{I}(S)$ satisfies the condition of state-dependent interpretations, i.e., $\mathsf{I}(S) \in \mathfrak{I}^{sd}$. In addition, we have $\mathit{size}(\mathsf{I}(S)) = i + \pi$. We define the size of the structure as $\mathit{size}(S) \stackrel{\text{def}}{=} i + \pi$.

Definition 13.5 (Minimality)**.** Let $S = (i, \pi, V_S, \prec_S)$ be a finite preferential structure and t be a time point s.t. $t \in [0, i+\pi[$. The set of preferred time points of t w.r.t. S, denoted by $\min_{\prec_S}(t)$, is defined as follows: $\min_{\prec_S}(t) \stackrel{\text{def}}{=} \{t' \in [\min_{<}\{t, i\}, i+\pi[\mid$ there is no $t'' \in [\min_{<}\{t, i\}, i+\pi[$ with $(V_S(t''), V_S(t')) \in \prec_S\}$.

The definition of minimality in our structures follows the principle of future preferred time points in the preferential interpretations. Given a $t \in [0, i+\pi[$, the set $\min_{\prec_S}(t)$ contains the minimal elements that come after t. Notice that in the case of $t \in [0, i[$, the minimal set starts with t and finishes with $i+\pi-1$. Whereas in the case of $t \in [i, i+\pi[$, we recall that in Definition 13.3 the interval $[i, i+\pi[$ is a finite representation of the final part of an interpretation where the elements within this interval are successors of each other, then the set $\min_{\prec_S}(t)$ contains all minimal elements of $[i, i+\pi[$.

Proposition 13.6. *Let $S = (i, \pi, V_S, \prec_S)$ be a structure, $\mathsf{I}(S) = (V, \prec)$ be its corresponding interpretation and $t, t', t_S, t'_S \in \mathbb{N}$ s.t.:*

$$t_S = \begin{cases} t & \textit{if } t < i; \\ i + (t-i) \bmod \pi & \textit{otherwise.} \end{cases} \qquad t'_S = \begin{cases} t' & \textit{if } t' < i; \\ i + (t'-i) \bmod \pi & \textit{otherwise.} \end{cases}$$

We have the following: $t' \in \min_{\prec}(t)$ iff $t'_S \in \min_{\prec_S}(t_S)$.

With the structures S introduced, we move to the procedure for checking the satisfiability of $\mathcal{L}_2$ sentences. We use a similar procedure to the one described in Section 11. Let $\alpha \in \mathcal{L}_2$ be a sentence, we define first the ordered set of sub-sentences of α_1.

Definition 13.7 (Sub-sentences)**.** Let $\alpha \in \mathcal{L}_2$, the set $Sf(\alpha)$ is recursively defined as follows:

- $Sf(p) := \{p\}$; $Sf(\neg p) := \{\neg p\}$;
- $Sf(\alpha_1 \wedge \alpha_2) := Sf(\alpha_1) \cup Sf(\alpha_2) \cup \{\alpha_1 \wedge \alpha_2\}$;
- $Sf(\alpha_1 \vee \alpha_2) := Sf(\alpha_1) \cup Sf(\alpha_2) \cup \{\alpha_1 \vee \alpha_2\}$;
- $Sf(\Box\alpha_1) := Sf(\alpha_1) \cup \{\Box\alpha_1\}$;
- $Sf(\Diamond\alpha_1) := Sf(\alpha_1) \cup \{\Diamond\alpha_1\}$;
- $Sf(\boxminus\alpha_1) := Sf(\alpha_1) \cup \{\boxminus\alpha_1\}$;
- $Sf(\diamondminus\alpha_1) := Sf(\alpha_1) \cup \{\diamondminus\alpha_1\}$.

Next, the labelling set $lab^S_\alpha(\cdot)$ is defined accordingly.

Definition 13.8 (Labelling sets)**.** Let $S = (i, \pi, V_S, \prec_S)$ be a finite preferential structure, $\alpha \in \mathcal{L}_2$ and $t \in [0, i+\pi[$. The set of sub-sentences of α that hold in t, denoted by $lab^S_\alpha(t)$, is defined as follows:

- $p \in lab^S_\alpha(t)$ iff $p \in V_S(t)$; $\neg\alpha_1 \in lab^S_\alpha(t)$ iff $\alpha_1 \notin lab^S_\alpha(t)$;
- $\alpha_1 \wedge \alpha_2 \in lab^S_\alpha(t)$ iff $\alpha_1, \alpha_2 \in lab^S_\alpha(t)$; $\alpha_1 \vee \alpha_2 \in lab^S_\alpha(t)$ iff $\alpha_1 \in lab^S_\alpha(t)$ or $\alpha_2 \in lab^S_\alpha(t)$;
- $\Diamond\alpha_1 \in lab^S_\alpha(t)$ iff $\alpha_1 \in lab^S_\alpha(t')$ for some $t' \in [\min_<\{t, i\}, i+\pi[$;
- $\Box\alpha_1 \in lab^S_\alpha(t)$ iff $\alpha_1 \in lab^S_\alpha(t')$ for all $t' \in [\min_<\{t, i\}, i+\pi[$;
- $\diamondminus\alpha_1 \in lab^S_\alpha(t)$ iff $\alpha_1 \in lab^S_\alpha(t')$ for some $t' \in \min_{\prec_S}(t)$;
- $\boxminus\alpha_1 \in lab^S_\alpha(t)$ iff $\alpha_1 \in lab^S_\alpha(t')$ for all $t' \in \min_{\prec_S}(t)$.

The set $lab^S_\alpha(t)$ contains the set of all sub-sentences of α that hold in t. For each sub-sentence, we can see that the labelling sets mimic the definition for that sentence's semantics. Moreover, we have the following property.

Proposition 13.9. *Given a finite preferential structure S and $\alpha \in \mathcal{L}_2$, we have $\mathsf{I}(S), 0 \models \alpha$ iff $\alpha \in lab^S_\alpha(0)$.*

We provide the generalised Lemma E.2 and its proof in Appendix E.

Checking the $\mathfrak{I}^{sd}$-satisfiability for $\mathcal{L}_2$ sentences uses the same procedure described in Section 11. Let α be a sentence in $\mathcal{L}_2$ and thanks to Theorem 12.1, if α is $\mathfrak{I}^{sd}$-satisfiable, then there exists an interpretation $I \in \mathfrak{I}^{sd}$ s.t. $size(I) \leq 2^{|\mathcal{P}|} \times |\alpha|$ that satisfies it. We make a non-deterministic guess for a structure S s.t. $size(S) \leq 2^{|\mathcal{P}|} \times |\alpha|$. Next, for each $\alpha_1 \in Sf(\alpha)$ in the increasing order of $|\alpha_1|$ and for each $t \in [0, i+\pi[$, we update $lab^S_\alpha(t)$ accordingly. At the end of this procedure, S is accepted as a structure for α if $\alpha \in lab^S_\alpha(0)$, otherwise, S is rejected.

Proposition 13.10. *Let $\alpha \in \mathcal{L}_2$. We have that α is $\mathfrak{I}^{sd}$-satisfiable iff there exists a finite preferential structure S such that $\mathsf{I}(S), 0 \models \alpha$ and $size(\mathsf{I}(S)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.*

Hence, to decide the satisfiability of a sentence $\alpha \in \mathcal{L}_2$, we can first guess a structure S bounded by $|\alpha| \times 2^{|\mathcal{P}|}$. Next, using the labelling function of S, we check the satisfiability of α by the UPI $\mathsf{I}(S)$.

Theorem 13.11. *$\mathfrak{I}^{sd}$-satisfiability problem for $\mathcal{L}_2$ sentences is* decidable.

Proof. Let $\alpha \in \mathcal{L}_2$. Thanks to Theorem 12.1, if α is $\mathfrak{I}^{sd}$-satisfiable, there exists a bounded interpretation I s.t. $I, 0 \models \alpha$ and $size(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$. We make a non-deterministic guess of a structure $S = (i, \pi, V_S, \prec_S)$ where $size(S) \leq |\alpha| \times 2^{|\mathcal{P}|}$ and use the labelling function $lab^S_\alpha(t)$ to check for all sub-sentences of α_1 in each $t \in [0, i]$. If $\alpha \in lab^S_\alpha(0)$, S is accepted as a structure and therefore α is satisfiable. Otherwise, S is rejected. Therefore, the $\mathfrak{I}^{sd}$-satisfiability for $\mathcal{L}_2$ sentences is a decidable problem. $\square$

14. Conclusion

We presented in this paper a new formalism called defeasible linear temporal logic ($LTL^{\sim}$) for reasoning about runs that have exceptional states in them. It extends also from the KLM approach to non-monotonic reasoning in order to handle exceptions. We adapted the notion of normality of *accessible worlds* in Britz and Varzinczak's (4) work on defeasible modal logic, to the notion of normality in a *run* in the case of temporal logic. By introducing non-monotonic temporal operators, defeasible properties that target the pertinent points of the execution can be expressed. Thus the elegant nature of LTL's vocabulary to express different properties of systems is preserved when they contain exceptional states.

In the study of the satisfiability checking problem, we establish a finite representation of models in the case of the fragments $\mathcal{L}_1$ and $\mathcal{L}_2$. We start by proving the bounded model properties (Sections 10 and 12) for both of them. Then in Sections 11 and 13, we define a compact model in the case of each of these sub-languages of $LTL^{\sim}$ respectively. While there is not a jump in the upper bound of the size of models when adding the $\mathrel{\Diamond\!\!\!\!\sim}$ operator in the case of $\mathcal{L}_1$ (the bound in $\mathcal{L}_1$ is the same as the fragment $L_{NNF}(\Diamond, \bigcirc)$ in Sistla and Clarke's work (19)), we notice an exponential blowup when adding the $\mathrel{\Box\!\!\!\!\sim}$ operator to the vocabulary in the case of $\mathcal{L}_2$ (for an input sentence α, the upper-bound is $|\alpha| \times 2^{|\mathcal{P}|}$ compared with $|\alpha|$ in the fragment $L(\Diamond)$ in Sistla and Clarke's work (19)).

The decidability of the satisfiability for the whole $\mathcal{L}^{\sim}$ language is still, as of yet, an open problem. In our investigation, when introducing $\bigcirc$ and $\mathcal{U}$ operator, the order of time points matters in the final part of preferential interpretation. Unlike the fragment $\mathcal{L}_2$, Lemmas 9.7 and 9.9 do not hold in the case of $\mathcal{L}^{\sim}$. Thus, in the same way as the fragment $L(\Diamond, \bigcirc)$ in Sistla and Clarke's work (19), we need to find a class of $LTL^{\sim}$ interpretations where interpretations can be ultimately periodic, in order to have the bounded-model property for $\mathcal{L}^{\sim}$. As it turns out, the $\mathfrak{I}^{sd}$ class of interpretations is not sufficient to prove the aforementioned property. We are currently investigating a new class of preferential interpretations. In this class, time points with the same set of sub-sentences express the same normality towards the other time points. We conjecture, using this class, that an ultimately periodic interpretation for $\mathcal{L}^{\sim}$ sentences can be induced and therefore have the decidability for the defeasible LTL language.

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Appendix A. Proofs of results in Section 9

Proposition 9.6. *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let $i \leq j \leq i' \leq j'$ be time points in final(I) s.t. $V(j) = V(j')$. Then we have $j \in \min_{\prec}(i)$ iff $j' \in \min_{\prec}(i')$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let us have four time points $i \leq j \leq i' \leq j' \in$ *final*(I).

- For the only-if part, we suppose that $j \in \min_{\prec}(i)$ and we prove that $j' \in \min_{\prec}(i')$. We have $i \leq i'$, $i' \leq j'$, $V(j) = V(j')$ and $j \in \min_{\prec}(i)$. Thanks to Proposition 8.3, $j' \in \min_{\prec}(i')$.
- For the if part, we suppose that $j' \in \min_{\prec}(i')$ and we prove that $j \in \min_{\prec}(i)$. We use a proof by contradiction. We assume that $j' \in \min_{\prec}(i')$ and we suppose that $j \notin \min_{\prec}(i)$. This implies that there exists $k \in [i, \infty[$ such that $(k, j) \in \prec$.
  - Case 1: $k \in [i', \infty[$. From Definition 8.1, since $V(j) = V(j')$ and $(k, j) \in \prec$, then $(k, j') \in \prec$ and therefore $j' \notin \min_{\prec}(i')$. This conflicts with our assumption that $j' \in \min_{\prec}(i')$.
  - Case 2: $k \in [i, i'[$. From Lemma 9.3, since $k \in$ *final*(I), there exists $k' \in [i', \infty[$ such that $V(k') = V(k)$. From Definition 8.1, since we have $V(j') = V(j)$, $V(k') = V(k)$ and $(k, j) \in \prec$, we also have $(k', j') \in \prec$, hence $j' \notin \min_{\prec}(i')$. This conflicts with the assumption that $j' \in \min_{\prec}(i')$.

□

Lemma 9.7. *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ be time points of final(I) where $V(i) = V(i')$. Then for every $\alpha \in \mathcal{L}_2$, we have $I, i \models \alpha$ iff $I, i' \models \alpha$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $i \leq i'$ in *final*(I) be such that $V(i) = V(i')$. We prove that $I, i \models \alpha$ iff $I, i' \models \alpha$ using structural induction on α.

- Base: $\alpha = p$. We know that $I, i \models p$ iff $p \in V(i)$. Since $V(i) = V(i')$, we have $p \in V(i')$. Thus $I, i' \models p$.
- $\alpha = \neg\alpha_1$. For the only-if part, we assume that $I, i \models \neg\alpha_1$ and suppose that $I, i' \not\models \neg\alpha_1$. $I, i' \not\models \neg\alpha_1$ implies that $I, i' \models \alpha_1$. Since the Lemma holds on α_1 and $I, i' \models \alpha_1$, then $I, i \models \alpha_1$. This conflicts with the assumption $I, i \models \neg\alpha_1$. We follow the same reasoning for the if part.
- $\alpha = \alpha_1 \wedge \alpha_2$. $I, i \models \alpha_1 \wedge \alpha_2$ means that $I, i \models \alpha_1$ and $I, i \models \alpha_2$. Since the Lemma holds on both α_1 and α_2, we have $I, i' \models \alpha_1$ and $I, i' \models \alpha_2$. Thus $I, i' \models \alpha_1 \wedge \alpha_2$.
- $\alpha = \Diamond\alpha_1$. For the only-if part, we assume that $I, i \models \Diamond\alpha_1$. This means that there exists $j \in [i, \infty[$ s.t. $I, j \models \alpha_1$. Thanks to Lemma 9.3, since $j \in$ *final*(I), then there exists $j' \in [i', \infty[$ where $V(j') = V(j)$. Thanks to the induction hypothesis, if $V(j) = V(j')$ and $I, j \models \alpha_1$, then $I, j' \models \alpha_1$. We conclude that $I, i' \models \Diamond\alpha_1$.

For the if part, we assume that $I, i' \models \Diamond\alpha_1$. This means that there is a $j' \in [i', \infty[$ s.t. $I, j' \models \alpha_1$. We know that $[i', \infty[\subseteq [i, \infty[$, and therefore we conclude that $I, i \models \Diamond\alpha_1$.

- $\alpha = \diamondsuit\alpha_1$. For the only-if part, we assume that $I, i \models \diamondsuit\alpha_1$. This means that there is a $j \in [i, \infty[$ s.t. $j \in \min_{\prec}(i)$ and $I, j \models \alpha_1$. Thanks to Lemma 9.3, since $j \in \mathit{final}(I)$, there exists $j' \in [i', \infty[$ such that $V(j') = V(j)$. Thanks to the induction hypothesis, if $V(j) = V(j')$ and $I, j \models \alpha_1$, then (I) $I, j' \models \alpha_1$. Thanks to Proposition 8.3, since $V(j) = V(j')$, $i \leq i'$, $i' \leq j'$ and $j \in \min_{\prec}(i)$, then we have (II) $j' \in \min_{\prec}(i')$. From (I) and (II), we conclude that $I, i' \models \diamondsuit\alpha_1$.

  For the if part, we assume that $I, i' \models \diamondsuit\alpha_1$. $I, i' \models \diamondsuit\alpha_1$ means that there is a $j' \in [i', \infty[$ such that $j' \in \min_{\prec}(i')$ and (I) $I, j' \models \alpha_1$. We need to prove that $j' \in \min_{\prec}(i)$. We suppose that $j' \notin \min_{\prec}(i)$. It means that there exists $k \in [i, \infty[$ such that $(k, j') \in \prec$. From Lemma 9.3, since $k \in \mathit{final}(I)$, there is $k' \in [i', \infty[$ such that $V(k) = V(k')$. By Definition 8.1, since $(k, j') \in \prec$ and $V(k') = V(k)$, we have $(k', j') \in \prec$ and therefore $j' \notin \min_{\prec}(i')$, conflicting with the assumption $j' \in \min_{\prec}(i')$. Thus, we have (II) $j' \in \min_{\prec}(i)$. From (I) and (II), we conclude that $I, i \models \diamondsuit\alpha$.

□

The proof of Lemma 9.9 can be found in Section B.

Proposition 9.12. *Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I. Then $N_1 \cup N_2$ is an acceptable sequence w.r.t. I s.t. $\mathit{size}(I, N_1 \cup N_2) \leq \mathit{size}(I, N_1) + \mathit{size}(I, N_2)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$, N_1, N_2 be two acceptable sequences w.r.t. I and let $I^{N_1} = (V^{N_1}, \prec^{N_1})$, $I^{N_2} = (V^{N_1}, \prec^{N_2})$ be two pseudo-interpretations over N_1 and N_2 respectively. We assume that $N = N_1 \cup N_2$.

We suppose that N is not an acceptable sequence w.r.t. I. It means that there are two time points $t, t' \in \mathit{final}(I)$ s.t. $V(t) = V(t')$ where $t \in N$ and $t' \notin N$. Since $t \in N$, t is either an element of N_1 or N_2. We assume that $t \in N_1$. By Definition 9.10, since $t \in N_1$ and N_1 is an acceptable sequence w.r.t. I, all time points of $\mathit{final}(I)$ that have the same valuation as t are in N_1. Since $t' \in \mathit{final}(I)$ and $V(t') = V(t)$, then $t' \in N_1$, and therefore $t' \in N$. This conflicts with the supposition of $t' \notin N$. Same reasoning applies if we take $t \in N_2$. We conclude that for all $t \in N$ s.t. $t \in \mathit{final}(I)$, all $t' \in \mathit{final}(I)$ s.t. $V(t') = V(t)$ are also in N. Thus, N is an acceptable sequence w.r.t. I.

In order to prove that $\mathit{size}(I, N) \leq \mathit{size}(I, N_1) + \mathit{size}(I, N_2)$, we need to prove that $\mathit{init}(I, N) \subseteq \mathit{init}(I, N_1) \cup \mathit{init}(I, N_2)$ and $\mathit{range}(I, N) \subseteq \mathit{range}(I, N_1) \cup \mathit{range}(I, N_2)$. Let $t \in N$ be a time point s.t. $t \in \mathit{init}(I, N)$. By the definition of $\mathit{init}(I, N)$, we know that $t \in \mathit{init}(I)$. Since N is a sequence containing only elements of N_1 or N_2, the time point t is either in N_1 or N_2. By definition of $\mathit{init}(I, N_1)$, if $t \in N_1$ and $t \in \mathit{init}(I)$, then $t \in \mathit{init}(I, N_1)$. The same goes in the case of $t \in N_2$. We conclude that if $t \in \mathit{init}(I, N)$, then $t \in \mathit{init}(I, N_1) \cup \mathit{init}(I, N_2)$.

Following the same line of thought, we can prove that $\mathit{final}(I, N) \subseteq \mathit{final}(I, N_1) \cup \mathit{final}(I, N_2)$ and consequently we can prove that $\mathit{range}(I, N) \subseteq \mathit{range}(I, N_1) \cup \mathit{range}(I, N_2)$.

Since $\mathit{init}(I^N) \subseteq \mathit{init}(I^{N_1}) \cup \mathit{init}(I^{N_2})$, we have $\mathit{length}(\mathit{init}(I^N)) \leq \mathit{length}(\mathit{init}(I^{N_1})) + \mathit{length}(\mathit{init}(I^{N_2}))$. Similarly, if $\mathit{range}(I^N) \subseteq \mathit{range}(I^{N_1}) \cup \mathit{range}(I^{N_2})$, then $\mathit{card}(\mathit{range}(I^N)) \leq \mathit{card}(\mathit{range}(I^{N_1})) + \mathit{card}(\mathit{range}(I^{N_2}))$. We conclude that $size(I^N) \leq size(I^{N_1}) + size(I^{N_2})$. $\square$

Proposition 9.13. *Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. If for all distinct $t, t' \in N$, we have $V(t') = V(t)$* only *when both $t, t' \in final(I, N)$, then $size(I, N) \leq 2^{|\mathcal{P}|}$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$ and N be an acceptable sequence w.r.t. I. We assume that for all $t, t' \in N$ s.t. we have $V(t') = V(t)$ only when both $t, t' \in final(N)$. Two cases are possible:

- $init(I, N)$ is empty. Since $card(range(I, N)) \leq 2^{|\mathcal{P}|}$, we have $size(I, N) \leq 2^{|\mathcal{P}|}$.
- $init(I, N)$ is not empty. We can see that for all $t \in init(I, N)$ and $t' \in N$ s.t. $t' \neq t$ we have $V(t') \neq V(t)$. If $init(I, N)$ has n time points having distinct valuations, then $range(final(I, N))$ has at most $2^{|\mathcal{P}|} - n$ valuations. Therefore, we have $size(I, N) \leq 2^{|\mathcal{P}|}$.

$\square$

Appendix B. Proofs of results for Lemma 9.9

NB: The results marked $(*)$ are introduced here, while they are omitted in the main text.

Proposition B.1 $(*)$**.** *Let $I = (V, \prec) \in \mathfrak{I}$ and $i \in final(I)$. For all $j \in final(I)$, there exists $j' \geq j$ such that $V(j') = V(i)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}$ and $i, j \in final(I)$. Let E be the set defined by $E = \{i' \in final(I) : V(i') = V(i)\}$. Since $i \in final(I)$, we have $E \neq \emptyset$. Suppose now that there does not exist $j' \geq j$ such that $V(j') = V(i)$. We have E is a non empty finite set of integers included in $[0, \ldots, j-1]$. Let $k = max\{k' \in E\}$. From the definitions of E and k, we have $k \in final(I)$ and there does not exist $k' > k$ such that $V(k') = V(k)$. This contradicts Lemma 9.3. We conclude that there exists $j' \geq j$ such that $V(j') = V(i)$. $\square$

Proposition B.2 $(*)$**.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over the same set of atomic propositions $\mathcal{P}$ s.t. $range(I) = range(I')$. For all $i \in final(I)$ and $i' \in final(I')$ such that $V(i) = V'(i')$, we have :*

(1) for all $j \in [i, \infty[$ there exists $j' \in [i', \infty[$ such that $V'(j') = V(j)$.
(2) for all $j \in \min_{\prec}(i)$ there exists $j' \in \min_{\prec'}(i')$ such that $V(j) = V'(j')$.

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ s.t. $range(I) = range(I')$ and let $i, i' \in final(I)$ be such that $V(i) = V'(i')$.

(1) Let $j \in [i, \infty[$. Since $i \in final(I)$, we have $j \in final(I)$. Moreover, given that $range(I) = range(I')$, we can assert that there exists $k \in final(I')$ such that $V'(k) = V(j)$. Hence, from Proposition B.1, there exists $j' \geq i'$ such that $V'(j') = V'(k) = V(j)$.
(2) Let $j \in \min_{\prec}(i)$. We have $j \in final(I)$. From Property (1) above, there exists $j' \geq i'$ such that $V'(j') = V(j)$. Suppose that $j' \notin \min_{\prec'}(i')$. Since $j' \geq i'$, there exists $k' \geq i'$ such that $(k', j') \in \prec'$. From Property (1) above, there exists $k \geq i$ such that $V(k) = V'(k')$. Since $V(k) = V'(k')$, $V'(j') = V(j)$, $(k', j') \in \prec'$

and, since I and I' are two faithful interpretations, we can assert that $(k, j) \in \prec$. Consequently, since $k \geq i$ and $(k, j) \prec$, we have $j \notin \min_{\prec}(i)$, which leads to a contradiction. We conclude that $j' \in \min_{\prec'}(i')$.

□

Proposition B.3 (*). *Let $\alpha \in \mathcal{L}_2$, $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over the same set of atomic propositions $\mathcal{P}$ s.t. $range(I) = range(I')$. For every $\alpha \in \mathcal{L}_2$ and every $i \in final(I)$ and $i' \in final(I')$ s.t. $V(i) = V'(i')$, we have:*

$$I, i \models \alpha \textit{ iff } I', i' \models \alpha.$$

Proof. Let $I = (V, \prec)$, $I' = (V', \prec')$ be two faithful interpretations belonging to $\mathfrak{I}^{sd}$ over the same set of atomic propositions $\mathcal{P}$ s.t. $range(I) = range(I')$. Let $\alpha \in \mathcal{L}_2$, and let $i \in final(I)$ and $i' \in final(I')$ be such that $V(i) = V'(i')$. Without loss of generality, we suppose that α does not contain $\vee$, $\Box$ and $\boxminus$. This proposition can be proven by induction on the structure of the sentence α.

- Base case : $\alpha = p$, with $p \in \mathcal{P}$. Since $V(i) = V'(i')$, we have $p \in V(i)$ iff $p \in V'(i')$, and thus $I, i \models p$ iff $I', i' \models p$.
- $\alpha = \neg\alpha_1$. By the induction hypothesis, $I, i \models \alpha_1$ iff $I', i' \models \alpha_1$. Hence, we have $I, i \not\models \alpha_1$ iff $I', i' \not\models \alpha_1$. It follows that $I, i \models \neg\alpha_1$ iff $I', i' \models \neg\alpha_1$.
- $\alpha = \alpha_1 \wedge \alpha_2$. We know that $I, i \models \alpha_1 \wedge \alpha_2$ holds iff $I, i \models \alpha_1$ and $I, i \models \alpha_2$. By the induction hypothesis, since $V(i) = V'(i')$, we have $I, i \models \alpha_1$ and $I, i \models \alpha_2$ iff $I', i' \models \alpha_1$ and $I', i' \models \alpha_2$. We conclude that $I, i \models \alpha_1 \wedge \alpha_2$ iff $I', i' \models \alpha_1 \wedge \alpha_2$.
- $\alpha = \Diamond\alpha_1$. First we prove that $I, i \models \Diamond\alpha_1$ implies that $I', i' \models \Diamond\alpha_1$. We assume that $I, i \models \Diamond\alpha_1$. It means that there exists $j \in [i, \infty[$ s.t. $I, j \models \alpha_1$. From Proposition B.2 (1), there exists $j' \in [i', \infty[$ such that $V'(j') = V(j)$. By the induction hypothesis, we have $I', j' \models \alpha_1$. Hence, we conclude that $I, i' \models \Diamond\alpha_1$. The if part can be proved with a similar reasoning.
- $\alpha = \diamondminus\alpha_1$. First we prove that $I, i \models \diamondminus\alpha_1$ implies that $I', i' \models \diamondminus\alpha_1$. We assume that $I, i \models \diamondminus\alpha_1$. Hence, there exists $j \in [i, \infty[$ s.t. $j \in \min_{\prec}(i)$ and $I, j \models \alpha_1$. From Proposition B.2 (2), there exists $j' \in \min_{\prec'}(i')$ such that $V'(j') = V(j)$. By the induction hypothesis, we have $I', j' \models \alpha_1$. We conclude that $I', i' \models \diamondminus\alpha_1$. The if part can be proved with a similar reasoning.

□

Corollary B.4 (*). *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over the same set of atomic propositions $\mathcal{P}$ s.t. $range(I) = range(I')$. For every $i \in final(I)$ and every $\alpha \in \mathcal{L}_2$, we have: if $I, i \models \alpha$, then there exists $i' \in final(I')$ s.t. $I', i' \models \alpha$.*

Proposition B.5 (*). *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ s.t. $init(I) \doteq init(I')$ and $range(I) = range(I')$. Then we have:*

$$\textit{For all } t, t' \in init(I),\ t' \in \min_{\prec}(t) \textit{ iff } t' \in \min_{\prec'}(t).$$

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $init(I) \doteq init(I')$ and $range(I) = range(I')$. Let $t, t' \in init(I)$ be such that $t' \in \min_{\prec}(t)$. Suppose that $t' \notin \min_{\prec'}(t)$. Since $t' \geq t$, there exists $t'' \geq t$ such that $(t'', t') \in \prec'$. There are two possible cases:

- $t'' \in init(I')$. Since $init(I) \doteq init(I')$, we have $V'(t'') = V(t'')$. Moreover, since I and I' are two faithful interpretations and $V'(t') = V(t')$, we have $(t'', t') \in \prec$. Since $t'' \geq t$, it follows that $t' \notin \min_{\prec}(t)$. This leads to a contradiction. We conclude that $t' \in \min_{\prec'}(t)$.
- $t'' \in final(I')$. Since $range(I) = range(I')$, there exists $t''' \in final(I)$ such that $V'(t'') = V(t''')$. Moreover, since I and I' are two faithful interpretations and $V'(t') = V(t')$, we have $(t''', t') \in \prec$. Since $t''' \geq t$, it follows that $t' \notin \min_{\prec}(t)$. This leads leads to a contradiction. We conclude that $t' \in \min_{\prec'}(t)$.

Same reasoning can be applied to prove the if part. □

Proposition B.6 (∗)**.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $init(I) \doteq init(I')$ and $range(I) = range(I')$. For all $t \in init(I)$ and $t' \in final(I)$ such that $t' \in \min_{\prec}(t)$, we have $\{t'' \in final(I') \mid V'(t'') = V(t')\} \subseteq \min_{\prec'}(t)$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $init(I) \doteq init(I')$ and $range(I) = range(I')$. Let $t \in init(I)$, $t' \in final(I)$, $t'' \in final(I')$ be such that $t' \in \min_{\prec}(t)$ and $V'(t'') = V(t')$. We show that $t'' \in \min_{\prec'}(t)$.

Suppose that $t'' \notin \min_{\prec'}(t)$. Since $t'' \geq t$, there exists $t''' \geq t$ such that $(t''', t'') \in \prec'$. There are two possible cases.

- $t''' \in init(I')$. Since $init(I) \doteq init(I')$, we have $V'(t''') = V(t''')$. Moreover, since I and I' are two faithful interpretations and $V'(t'') = V(t')$, we have $(t''', t') \in \prec$. Since $t''' \geq t$, it follows that $t' \notin \min_{\prec}(t)$. This leads to a contradiction. We conclude that $t'' \in \min_{\prec'}(t)$.
- $t''' \in final(I')$. Since $range(I) = range(I')$, there exists $u \in final(I)$ such that $V'(t''') = V(u)$. Moreover, since I and I' are two faithful interpretations and $V'(t'') = V(t')$, we have $(u, t') \in \prec$. Since $u \geq t$, it follows that $t' \notin \min_{\prec}(t)$. There is a contradiction. We conclude that $t'' \in \min_{\prec'}(t)$.

□

Lemma B.7 (∗)**.** *Let $I = (V, \prec) \in \mathfrak{I}^{sd}$ and $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $V'(0) = V(0)$, $init(I) \doteq init(I')$, and $range(I) = range(I')$. Then for all $\alpha \in \mathcal{L}_2$, we have :*

$$\textit{For all } t \in init(I) \cup \{0\},\ I, t \models \alpha \textit{ iff } I', t \models \alpha.$$

The singleton $\{0\}$ is there in case of an empty $init(I)$.

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, $I' = (V', \prec') \in \mathfrak{I}^{sd}$ be two faithful interpretations over $\mathcal{P}$ such that $V'(0) = V(0)$, $init(I) \doteq init(I')$, and $range(I) = range(I')$. Let $\alpha \in \mathcal{L}_2$ and $t \in init(I) \cup \{0\}$. Without loss of generality, we suppose that α does not contain $\vee$, $\Box$ and $\boxslash$.

First, notice that in the case where $init(I)$ and $init(I')$ are empty intervals, we necessarily have $t = 0$. Moreover, since $t \in final(I)$ and $t \in final(I')$ and $V(0) = V'(0)$, from Proposition B.3, we can assert that $I, t \models \alpha$ iff $I', t \models \alpha$. Consequently, the property to be proved is true. Now, we suppose that $init(I)$ and $init(I')$ are non empty intervals. Hence, we have $t \in init(I)$ and $t \in init(I')$. We prove that $I, t \models \alpha$ iff $I', t \models \alpha$ by structural induction on α.

- Base case : $\alpha = p$. Since $t \in \mathit{init}(I)$, we have $V(t) = V'(t)$. Hence, $p \in V(t)$ iff $p \in V'(t)$. Thus $I, t \models p$ iff $I', t \models p$.
- $\alpha = \neg\alpha_1$. By the induction hypothesis, we have $I, t \models \alpha_1$ iff $I', t \models \alpha_1$. Hence, it is not the case that $I, t \models \alpha_1$ iff it is not the case that $I', t \models \alpha_1$. We conclude that, $I, t \models \neg\alpha_1$ iff $I', t \models \neg\alpha_1$.
- $\alpha = \alpha_1 \wedge \alpha_2$. We have $I, t \models \alpha_1 \wedge \alpha_2$ iff $I, t \models \alpha_1$ and $I, t \models \alpha_2$. Using the induction hypothesis, it follows that $I, t \models \alpha_1$ and $I, t \models \alpha_2$ iff $I', t \models \alpha_1$ and $I', t \models \alpha_2$. We conclude that $I, t \models \alpha_1 \wedge \alpha_2$ iff $I', t \models \alpha_1 \wedge \alpha_2$.
- $\alpha = \Diamond\alpha_1$. Suppose that $I, t \models \Diamond\alpha_1$. There exists a $t' \in [t, \infty[$ s.t. $I, t' \models \alpha_1$. Two cases are possible w.r.t. t'.
  - $t' \in \mathit{init}(I)$. By the induction hypothesis, we have $I', t' \models \alpha_1$. Hence, we conclude that $I', t \models \Diamond\alpha_1$.
  - $t' \in \mathit{final}(I)$. Since $\mathit{range}(I) = \mathit{range}(I')$, there exists $t'' \in \mathit{final}(I')$ such that $V'(t'') = V(t')$. From Proposition B.3, we have $I', t'' \models \alpha_1$. Since $t'' > t$, we have $I', t \models \Diamond\alpha_1$.

  Same reasoning can be applied to prove the if part.
- $\alpha = \Diamond\!\!\!\!\sim\alpha_1$. Suppose that $I, t \models \Diamond\!\!\!\!\sim\alpha_1$. There exists $t' \in \min_{\prec}(t)$ s.t. $I, t' \models \alpha_1$. Two cases are possible w.r.t. t'.
  - $t' \in \mathit{init}(I)$. By the induction hypothesis, we have $I', t' \models \alpha_1$. Moreover, from Proposition B.5, we have $t' \in \min_{\prec'}(t)$. Hence, we conclude that $I', t \models \Diamond\!\!\!\!\sim\alpha_1$.
  - $t' \in \mathit{final}(I)$. Since $\mathit{range}(I) = \mathit{range}(I')$, there exists $t'' \in \mathit{final}(I')$ such that $V'(t'') = V(t')$. From Proposition B.3, we have $I', t'' \models \alpha_1$. From Proposition B.6, we have $t'' \in \min_{\prec'}(t)$. Hence, we conclude that $I', t \models \Diamond\!\!\!\!\sim\alpha_1$. Same reasoning can be applied to prove the if part.

$\square$

Lemma 9.9 is a direct result of result of Lemma B.7.

Appendix C. Proofs of results in Section 11

Lemma 11.5. *Let $\alpha \in \mathcal{L}_1$ be an $\mathfrak{I}$-satisfiable sentence and $I = (V, \prec) \in \mathfrak{I}$ be an interpretation such that $I, 0 \models \alpha$. Let I^N be the pseudo-interpretation of I over the* finite *sequence N such that $I^N, 0 \models_{\mathscr{P}} \alpha$, and $I' = (V', \prec')$ be the induced interpretation from I^N. Let $S = (n, V_S, \prec_S)$ be the finite preferential structure where $n = |N|$, $V_S(t) = V'(t)$ for each $t \in [0, |N| - 1]$ and $\prec_S = \prec'$. Let $\mathsf{I}(S) = (V'', \prec'')$ be the induced interpretation from S. We have the following:*

- *$\prec'' = \prec'$ and $V''(t) = V'(t)$ for each $t \in \mathbb{N}$;*
- *for every $\alpha_1 \in \mathit{Sf}(\alpha)$, we have $\alpha_1 \in \mathit{lab}^S_\alpha(t)$ iff $\mathsf{I}(S), t \models \alpha_1$.*

Proof. Let $\alpha \in \mathcal{L}_1$ be an $\mathfrak{I}$-satisfiable sentence and $I = (V, \prec) \in \mathfrak{I}$ be an interpretation such that $I, 0 \models \alpha$. Let $I^N = (V^N, \prec^N)$ be the pseudo-interpretation of I over the *finite* sequence N such that $I^N, 0 \models_{\mathscr{P}} \alpha$, and $I' = (V', \prec')$ be the induced interpretation from I^N. Let $S = (n, V_S, \prec_S)$ be the finite preferential structure where $n = |N|$, $V_S(t) = V'(t)$ for each $t \in [0, |N| - 1]$ and $\prec_S = \prec'$. Let $\mathsf{I}(S) = (V'', \prec'')$ be the induced interpretation from S.

By Definition 11.2, we have $\prec'' = \prec_S$, $V''(t) = V_S(t)$ for each $t \in [0, n-1]$ and $V''(t) = V''(n-1)$ for each $t \in [n, \infty[$. Since $\prec' = \prec_S$, $V'(t) = V_S(t)$ for each $t \in [0, |N| - 1]$,

$V'(t) = V'(|N|-1)$ for each $t \in [|N|, \infty[$ (see Defnition 10.4) and $n = |N|$, then $\prec''=\prec'$ and $V''(t) = V'(t)$ for each $t \in \mathbb{N}$. Therefore, the first item of the lemma holds. As such, we shall the interpretations I' and $\mathsf{I}(S)$ interchangeably.

Going back to the pseudo-interpretation I^N, we know that N is a finite sequence. By the definition of truth values of sentences in pseudo-interpretations (after Definition 9.2), we have the following:

- for any $\bigcirc\alpha_1$ such that $I^N, t_i \models_{\mathscr{P}} \bigcirc\alpha_1$, we have $t_i + 1 \in N$ and $I^N, t_i + 1 \models_{\mathscr{P}} \alpha_1$;
- for any $\Diamond\alpha_1$ such that $I^N, t_i \models_{\mathscr{P}} \Diamond\alpha_1$, there exists $t_j \in N$ such that $t_j \geq t_i$ and $I^N, t_j \models_{\mathscr{P}} \alpha_1$;
- for any $\mathrel{\Diamond\!\!\!\!\!\sim}\alpha_1$ such that $I^N, t_i \models_{\mathscr{P}} \mathrel{\Diamond\!\!\!\!\!\sim}\alpha_1$, there exists $t_j \in N$ such that $t_j \in \min_{\prec^N}(t_i)$ and $I^N, t_j \models_{\mathscr{P}} \alpha_1$.

Since I' is the induced from I^N, the aforementioned properties hold for I' as well. Given $i, j \in [0, |N|-1]$, we have the following:

- (I) for any $\bigcirc\alpha_1$ such that $I', i \models \bigcirc\alpha_1$, we have $i+1 \in [0, |N|-1]$ and $I', i+1 \models \alpha_1$;
- (II) for any $\Diamond\alpha_1$ such that $I', i \models \Diamond\alpha_1$, there exists $j \in [0, |N|-1]$ such that $j \geq i$ and $I', j \models \alpha_1$;
- (III) for any $\mathrel{\Diamond\!\!\!\!\!\sim}\alpha_1$ such that $I', i \models \mathrel{\Diamond\!\!\!\!\!\sim}\alpha_1$, there exists $j \in [0, |N|-1]$ such that $\in \min_{\prec'}(i)$ and $I', j \models \alpha_1$.

It is important to note that these properties hold for $\mathsf{I}(S)$ as well. In other words, any sub-sentence of the form $\bigcirc\alpha_1$, $\Diamond\alpha_1$ or $\mathrel{\Diamond\!\!\!\!\!\sim}\alpha_1$ that holds at $t \in [0, n-1]$ is satisfied within the finite part of $\mathsf{I}(S)$. Moving on to the second item of the lemma, let $\alpha_1 \in \mathit{Sf}(\alpha)$ and $t \in [0, n-1]$. We use structural induction on sub-sentences of α_1.

- $\alpha_1 = p$. By Definition 11.2, since $\mathsf{I}(S)$ is the S-induced interpretation and $t \in [0, n-1]$, then $V''(t) = V_S(t)$. Thus, we have $p \in V_S(t)$ iff $p \in V''(t)$. Therefore, we have $p \in \mathit{lab}_\alpha^S(t)$ iff $\mathsf{I}(S), t \models p$.
- $\alpha_1 = \neg p$. Following the same reasoning as in the case of p, we have $V''(t) = V_S(t)$. Thus, we have $p \notin V_S(t)$ iff $p \notin V''(t)$. Therefore, we have $\neg p \in \mathit{lab}_\alpha^S(t)$ iff $\mathsf{I}(S), t \models \neg p$.
- $\alpha_1 = \alpha_2 \wedge \alpha_3$. Assume that $\alpha_2 \wedge \alpha_3 \in \mathit{lab}_\alpha^S(t)$, we have $\alpha_2, \alpha_3 \in \mathit{lab}_\alpha^S(t)$. By the induction hypothesis, $\alpha_2, \alpha_3 \in \mathit{lab}_\alpha^S(t)$ iff $\mathsf{I}(S), t \models \alpha_2$ and $\mathsf{I}(S), t \models \alpha_3$. Therefore, we have $\alpha_2 \wedge \alpha_3 \in \mathit{lab}_\alpha^S(t)$ iff $\mathsf{I}(S), t \models \alpha_2 \wedge \alpha_3$.
- $\alpha_1 = \alpha_2 \vee \alpha_3$. Assume that $\alpha_2 \vee \alpha_3 \in \mathit{lab}_\alpha^S(t)$, we either have $\alpha_2 \in \mathit{lab}_\alpha^S(t)$ or $\alpha_2 \in \mathit{lab}_\alpha^S(t)$. Assume that $\alpha_2 \in \mathit{lab}_\alpha^S(t)$, by the induction hypothesis, $\alpha_2 \in \mathit{lab}_\alpha^S(t)$ iff $\mathsf{I}(S), t \models \alpha_2$. Thus, we have $\alpha_2 \vee \alpha_3 \in \mathit{lab}_\alpha^S(t)$ iff $\mathsf{I}(S), t \models \alpha_2 \vee \alpha_3$. Same reasoning can be applied in the case of $\alpha_3 \in \mathit{lab}_\alpha^S(t)$.
- $\alpha_1 = \Diamond\alpha_2$.
  - For the only-if part, we assume that $\Diamond\alpha_2 \in \mathit{lab}_\alpha^S(t)$, we have $\alpha_2 \in \mathit{lab}_\alpha^S(t')$ where $t' \in [t, n-1]$. By the induction hypothesis, since $\alpha_2 \in \mathit{lab}_\alpha^S(t')$ and $t' \in [0, n-1]$, then we have $\alpha_2 \in \mathit{lab}_\alpha^S(t')$ iff $\mathsf{I}(S), t' \models \alpha_2$. Therefore, we have $\mathsf{I}(S), t \models \Diamond\alpha_2$.
  - For the if part, we assume that $\mathsf{I}(S), t \models \Diamond\alpha_2$. Knowing $\mathsf{I}(S)$ is the same as I', since $t \in [0, n-1]$ and thanks to item (II), then there is $t' \in [t, n-1]$ where $\mathsf{I}(S), t' \models \alpha_2$. By the induction hypothesis, since $\mathsf{I}(S), t' \models \alpha_2$ and $t' \in [t, n-1]$, then we have $\alpha_2 \in \mathit{lab}_\alpha^S(t')$. Therefore, we have $\Diamond\alpha_2 \in \mathit{lab}_\alpha^S(t)$.
- $\alpha_1 = \bigcirc\alpha_2$. Assume that $\bigcirc\alpha_2 \in \mathit{lab}_\alpha^S(t)$, we have $\alpha_2 \in \mathit{lab}_\alpha^S(t+1)$ where $t + 1 \leq n - 1$ (thanks to item (I), there is no need to check the case of $t = n - 1$).

By the induction hypothesis, since $\alpha_2 \in lab^S_\alpha(t+1)$ and $t+1 \in [0, n-1]$, we have $\alpha_2 \in lab^S_\alpha(t+1)$ iff $\mathsf{I}(S), t+1 \models \alpha_2$. Therefore, we have $\bigcirc\alpha_2 \in lab^S_\alpha(t)$ iff $\mathsf{I}(S), t \models \bigcirc\alpha_2$.

- $\alpha_1 = \Box\alpha_{bool}$. Assume that $\Box\alpha_{bool} \in lab^S_\alpha(t)$, we have $\alpha_{bool} \in lab^S_\alpha(t')$ for all $t' \in [t, n-1]$. By the induction hypothesis, since we have $\alpha_{bool} \in lab^S_\alpha(t')$ for each $t' \in [t, n-1]$, then for all $t' \in [t, n-1]$, we have (i) $\alpha_{bool} \in lab^S_\alpha(t')$ iff $\mathsf{I}(S), t' \models \alpha_{bool}$. Moreover, we have $V''(t') = V_S(n-1) = V(n-1)$ for all $t' \in [n, \infty[$. Since α_{bool} is a Boolean sentence, $\mathsf{I}(S), n-1 \models \alpha_{bool}$ and $V''(t') = V''(n-1)$ for all $t' \in [i, \infty[$; then we deduce that (ii) $\mathsf{I}(S), n-1 \models \alpha_{bool}$ iff $\mathsf{I}(S), t' \models \alpha_{bool}$ for all $t' \in [n, \infty[$. From (i) and (ii), we conclude that $\Box\alpha_{bool} \in lab^S_\alpha(t)$ iff $\mathsf{I}(S), t \models \Box\alpha_{bool}$.
- $\alpha_1 = \diamondsuit\alpha_2$.
  - For the only-if part, we assume that $\diamondsuit\alpha_2 \in lab^S_\alpha(t)$. We have $\alpha_2 \in lab^S_\alpha(t')$ where $t' \in \min_{\prec_S}(t)$. By the induction hypothesis, since $\alpha_2 \in lab^S_\alpha(t')$ and $t' \in [0, n-1]$, then we have (i) $\mathsf{I}(S), t' \models \alpha_2$. Moreover, since $t' \in \min_{\prec_S}(t)$, we have (ii) $t' \in \min_{\prec''}(t)$. From (i) and (ii), we conclude that $\mathsf{I}(S), t \models \diamondsuit\alpha_2$.
  - For the if part, we assume that $\mathsf{I}(S), t \models \diamondsuit\alpha_2$. Knowing $\mathsf{I}(S)$ is the same as I', since $t \in [0, n-1]$ and thanks to item (III), then there is $t' \in [t, n-1]$ such that $t' \in \min_\prec(t)$ and $\mathsf{I}(S), t' \models \alpha_2$. By the induction hypothesis, since $t' \in [t, n-1]$ and $\mathsf{I}(S), t' \models \alpha_2$, then we have (i) $\alpha_2 \in lab^S_\alpha(t')$. Moreover, since $t' \in \min_{\prec''}(t)$, we have (ii) $t' \in \min_{\prec_S}(t)$. From (i) and (ii), we conclude that $\diamondsuit\alpha_2 \in lab^S_\alpha(t)$.

$\square$

Appendix D. Proofs of results in Section 12

Lemma 12.10. *Let $\alpha_1 \in \mathcal{L}_2$ be a sentence, $I = (V, \prec) \in \mathfrak{I}^{sd}$ and let T be a non-empty acceptable sequence w.r.t. I where for all $t_i \in T$ we have $I, t_i \models \diamondsuit\alpha_1$. Then for all $t, t' \in Anchors(I, T, \diamondsuit\alpha_1)$ s.t. $V(t) = V(t')$ and $t \neq t'$, we have $t, t' \in final(I, Anchors(I, T, \diamondsuit\alpha_1))$.*

Proof. Let $\alpha_1 \in \mathcal{L}_2$, let T be a non-empty acceptable sequence w.r.t. $I \in \mathfrak{I}^{sd}$ where for all $t_i \in T$ we have $I, t_i \models \diamondsuit\alpha_1$. Just as a reminder, we have $Anchors(I, T, \diamondsuit\alpha_1) = \bigcup_{t_i \in T} ST(I, AS(I, \min_\prec(t_i)), \alpha_1)$. Thus, there exists $t_i, t'_i \in T$ such that $t \in ST(I, AS(I, \min_\prec(t_i)), \alpha_1)$ and $t' \in ST(I, AS(I, \min_\prec(t'_i)), \alpha_1)$. Suppose that the lemma is false. Then there are two time points $t, t' \in Anchors(I, T, \diamondsuit\alpha_1)$ with $t \neq t'$ such that t is in $init(I, Anchors(I, T, \diamondsuit\alpha_1))$ at least and $V(t) = V(t')$. Note that $t \in init(I)$, since we have $t \in init(I, Anchors(I, T, \diamondsuit\alpha_1))$. Without loss of generality, we assume that $t < t'$. From Definition 12.6, we have $t \in AS(I, (\mathsf{t}_{\alpha_1}^{I, AS(I, \min_\prec(t_i))}))$ where $\mathsf{t}_{\alpha_1}^{I, AS(I, \min_\prec(t_i))}$ is the chosen occurrence that satisfies α_1 in $AS(I, \min_\prec(t_i))$. Thanks to Definitions 12.2 and 12.4, since $t \in init(I)$, we can see that: $t = \mathsf{t}_{\alpha_1}^{I, AS(I, \min_\prec(t_i))}$. Moreover, (1) there is no $t'' \in final(I, AS(I, \min_\prec(t_i)))$ s.t. $I, t'' \models \alpha_1$ and (2) $t = \mathsf{t}_{\alpha_1}^{I, AS(I, \min_\prec(t_i))} = max_<\{t'' \in init(I, AS(I, \min_\prec(t_i))) \mid I, t'' \models \alpha_1\}$ (see Definition 12.4). On the other hand, thanks to Proposition 8.3, since $t < t'$, $V(t) = V(t')$ and $t \in \min_\prec(t_i)$, we have $t' \in \min_\prec(t_i)$. Hence, we have $t' \in AS(I, \min_\prec(t_i))$. Since

$t' \in ST(I, AS(I, \min_{\prec}(t'_i)), \alpha_1)$, we also have $I, t' \models \alpha_1$. From this and the property (1), we can assert that t' does not belong to $\mathit{final}(I, AS(I, \min_{\prec}(t_i)))$. It follows that $t' \in \mathit{init}(I, AS(I, \min_{\prec}(t_i)))$. From the property (2) we can assert that $t \geq t'$, which leads to a contradiction since $t < t'$. Therefore, for all $t, t' \in \mathit{Anchors}(I, T, \diamondminus\alpha_1)$ s.t. $V(t) = V(t')$, we must have $t, t' \in \mathit{final}(\mathit{Anchors}(I, T, \diamondminus\alpha_1))$. $\square$

Proposition 12.11. *Let $\alpha \in \mathcal{L}_2$ be of the form $\mathcal{O}\alpha_1$, where $\mathcal{O} \in \{\Diamond, \Box, \diamondminus, \boxminus\}$ and $\alpha_1 \in \mathcal{L}_2$. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I where for all $t \in T$ we have $I, t \models \alpha$. Then, we have:*

$$\mathit{size}(I, \mathit{Anchors}(I, T, \alpha)) \leq 2^{|\mathcal{P}|}.$$

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. We show that is the case for the temporal operators $\Box, \Diamond, \boxminus, \diamondminus$:

- Since $\mathit{size}(I, \mathit{Anchors}(I, T, \Box\alpha_1)) = \mathit{size}(I, \emptyset) = 0$, the result follows immediately.
- Since $\mathit{size}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1)) = \mathit{size}(I, ST(I, \mathbb{N}, \alpha_1)) = 1$, the result follows immediately.
- T is an acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \diamondminus\alpha_1$. From Proposition 12.10, for all $t'_i, t'_j \in \mathit{Anchors}(I, T, \diamondminus\alpha_1)$ s.t. $V(t'_i) = V(t'_j)$ and $t'_i \neq t'_j$, we have $t'_i, t'_j \in \mathit{final}(I, \mathit{Anchors}(I, T, \diamondminus\alpha_1))$. Thanks to Proposition 9.13, we conclude that $\mathit{size}(\mathit{Anchors}(I, T, \diamondminus\alpha_1)) \leq 2^{|\mathcal{P}|}$.
- Going back to Definition 12.9, we have $\mathit{Anchors}(I, T, \boxminus\alpha_1) = \mathit{DR}(I, \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i)))$. We denote the acceptable sequence $\bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$ by N. From Definition 12.8 we have $\mathit{Anchors}(I, T, \boxminus\alpha_1) = \mathit{DR}(I, N) = \bigcup_{v \in \mathit{val}(I,N)} ST(I, N, \alpha_v)$. Moreover, we know that $\mathit{size}(ST(I, N, \alpha_v)) = 1$ for all $v \in \mathit{val}(I, N)$. Consequently, thanks to Proposition 9.12, we have $\mathit{size}(\bigcup_{v \in \mathit{val}(I,N)} ST(I, N, \alpha_v)) \leq \mathit{card}(\mathit{val}(I, N))$. We can see that $\mathit{card}(\mathit{val}(I, N)) \leq 2^{|\mathcal{P}|}$, and therefore $\mathit{size}(\mathit{Anchors}(I, T, \boxminus\alpha_1)) = \mathit{size}(\bigcup_{v \in \mathit{val}(I,N)} ST(I, N, \alpha_v)) \leq 2^{|\mathcal{P}|}$.

$\square$

Proposition 12.12. *Let $\alpha_1 \in \mathcal{L}_2$, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \boxminus\alpha_1$, with $\alpha_1 \in \mathcal{L}_2$. For all acceptable sequences N w.r.t. I s.t. $\mathit{Anchors}(I, T, \boxminus\alpha_1) \subseteq N$ and for all $t_i \in N \cap T$, let $I^N = (V^N, \prec^N)$ be the pseudo-interpretation over N and $t' \in N$. We have the following:*

$$\text{If } t' \notin \min_{\prec}(t_i), \text{ then } t' \notin \min_{\prec^N}(t_i).$$

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \boxminus\alpha_1$, with $\alpha_1 \in \mathcal{L}_2$. Let N be an acceptable sequence w.r.t. I s.t. $\mathit{Anchors}(I, T, \boxminus\alpha_1) \subseteq N$. Let $t_i \in N \cap T$. Let $t' \in N$ be a time point s.t. $t' \notin \min_{\prec}(t_i)$. There are two possible cases:

- $t' \notin [t_i, \infty[$: Since $t' \notin [t_i, \infty[$, then $t' \notin [t_i, \infty[\cap N$. Therefore, we conclude that $t' \notin \min_{\prec^N}(t_i)$.
- $t' \in [t_i, \infty[$: Since $\prec$ satisfies the well-foundedness condition (that is why T must not be empty), $t' \notin \min_{\prec}(t_i)$ implies that there exists a time point $t'' \in \min_{\prec}(t_i)$

s.t. $(t'', t') \in \prec$. Let $\alpha_{t''}$ be the representative sentence of $V(t'')$ (recall that $\alpha_{t''} = \bigwedge\{p \mid p \in V(t'')\} \wedge \bigwedge\{\neg p \mid p \notin V(t'')\}$). For the sake of readability, we shall denote the sequence $\bigcup_{t \in T} AS(I, \min_{\prec}(t))$ with M. Notice that there exists $V \in \mathit{val}(I, M)$ such that $V = V(t'')$ since $t_i \in T$ and $t'' \in \min_{\prec}(t_i)$. Thanks to Definition 12.8, since $DR(I, M) = \bigcup_{v \in \mathit{val}(I,M)} ST(I, M, \alpha_v)$ and $V(t'') \in \mathit{val}(I, M)$, we can find $t''' \in ST(I, M, \alpha_{t''})$ where $t''' \in DR(I, M) \subseteq N$, $V(t''') = V$ and $t''' \geq t''$. Since $(t'', t') \in \prec$, $I \in \mathfrak{I}^{sd}$ and $V(t''') = V(t'')$, we have $(t''', t') \in \prec$. Moreover, we have $t''', t' \in N$, and therefore $(t''', t') \in \prec^N$. Since $t''' \in [t_i, \infty[\cap N$ and $(t''', t') \in \prec^N$, we conclude that $t' \notin \min_{\prec^N}(t_i)$.

$\square$

Proposition 12.14. *Let $\alpha \in \mathcal{L}_2$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. Then, we have $\mathit{size}(I, \mathit{Keep}(I, T, \alpha)) \leq \mu(\alpha) \times 2^{|\mathcal{P}|}$.*

Proof. Let $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$ where $\alpha \in \mathcal{L}_2$.

We use structural induction on T and α in order to prove this property.

- Base: $\alpha = p$ or $\alpha = \neg p$. We have $\mathit{Keep}(I, T, \alpha) = \emptyset$. Since $\mathit{size}(I, \emptyset) = 0 \leq \mu(\alpha) \times 2^{|\mathcal{P}|} = 0$, then the property holds on atomic propositions.
- $\alpha = \alpha_1 \wedge \alpha_2$. Since $I, t \models \alpha_1 \wedge \alpha_2$ for all $t \in T$, we can assert that $I, t \models \alpha_1$ and $I, t \models \alpha_2$. By applying the induction hypothesis on T, α_1 and α_2, we have $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$ and $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_2)) \leq \mu(\alpha_2) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 9.12, $\mathit{size}(\mathit{Keep}(I, T, \alpha_1 \wedge \alpha_2)) \leq (\mu(\alpha_1) + \mu(\alpha_2)) \times 2^{|\mathcal{P}|}$. We conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_1 \wedge \alpha_2)) \leq (\mu(\alpha_1 \wedge \alpha_2)) \times 2^{|\mathcal{P}|}$.
- $\alpha = \alpha_1 \vee \alpha_2$. Since $I, t \models \alpha_1 \vee \alpha_2$ for all $t \in T$, we have $I, t \models \alpha_1$ or $I, t \models \alpha_2$. Let T_1 be the sequence (resp. T_2) containing all $t_1 \in T$ (resp.$t_2 \in T$) s.t. $I, t_1 \models \alpha_1$ (resp. $I, t_2 \models \alpha_2$). Using induction hypothesis on T_1, T_2, α_1 and α_2, we have $\mathit{size}(I, \mathit{Keep}(I, T_1, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$ and $\mathit{size}(I, \mathit{Keep}(I, T_2, \alpha_2)) \leq \mu(\alpha_2) \times 2^{|\mathcal{P}|}$. We conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_1 \vee \alpha_2)) \leq (\mu(\alpha_1 \vee \alpha_2)) \times 2^{|\mathcal{P}|}$ in the same way as $\alpha_1 \wedge \alpha_2$.
- $\alpha = \Diamond\alpha_1$. First of all, we proved in Proposition 12.11 that (I) $\mathit{size}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1)) \leq 2^{|\mathcal{P}|}$. On the other hand, thanks to Definition 12.9 it is easy to see that $\mathit{size}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1))$ is a non-empty acceptable sequence w.r.t. I s.t. for all $t' \in \mathit{Anchors}(I, T, \Diamond\alpha_1)$ we have $I, t' \models \alpha_1$. By the induction hypothesis on $\mathit{Anchors}(I, T, \Diamond\alpha_1)$ and α_1, we have (II) $\mathit{size}(I, \mathit{Keep}(I, \mathit{Anchors}(I, T, \Diamond\alpha_1), \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 9.12, from (I) and (II) we conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \Diamond\alpha_1)) \leq (1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\Diamond\alpha_1) \times 2^{|\mathcal{P}|}$.
- $\alpha = \Box\alpha_1$. As a result of semantics of the $\Box$ operator, we can see that for all $t \in T$ we have $I, t \models \alpha_1$. By the induction hypothesis on T and α_1, we have $\mathit{size}(I, \mathit{Keep}(I, T, \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Since $\mathit{Keep}(I, T, \alpha_1) = \mathit{Keep}(I, T, \Box\alpha_1)$ then $\mathit{size}(I, \mathit{Keep}(I, T, \Box\alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. We conclude that $\mathit{size}(I, \mathit{Keep}(I, T, \Box\alpha_1)) \leq \mu(\Box\alpha_1) \times 2^{|\mathcal{P}|}$.
- $\alpha = \tilde{\Diamond}\alpha_1$. First of all, we proved in Proposition 12.11 that (I) $\mathit{size}(I, \mathit{Anchors}(I, T, \tilde{\Diamond}\alpha_1)) \leq 2^{|\mathcal{P}|}$. On the other hand, thanks to Definition 12.9, it is easy to see that $\mathit{Anchors}(I, T, \tilde{\Diamond}\alpha_1)$ is a non-empty acceptable sequence w.r.t. I s.t. for all $t' \in \mathit{Anchors}(I, T, \tilde{\Diamond}\alpha_1)$ we have $I, t' \models \alpha_1$. By the induction hypothesis on $\mathit{Anchors}(I, T, \tilde{\Diamond}\alpha_1)$ and α_1, we have (II)

$size(I, Keep(I, Anchors(I, T, \tilde{\Diamond}\alpha_1), \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 9.12, from (I) and (II), we conclude that $size(I, Keep(I, T, \tilde{\Diamond}\alpha_1)) \leq (1+\mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\tilde{\Diamond}\alpha_1) \times 2^{|\mathcal{P}|}$.
- $\alpha = \tilde{\Box}\alpha_1$. First of all, we proved in Proposition 12.11 that (I) $size(I, Anchors(I, T, \tilde{\Box}\alpha_1)) \leq 2^{|\mathcal{P}|}$. On the other hand, from Definition 12.13, we have $T' = \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$. It is easy to see that for all $t' \in T'$ we have $I, t' \models \alpha_1$ and that T' is a non-empty acceptable sequence w.r.t. I. By the induction hypothesis on T' and α_1, we have (II) $size(I, Keep(I, T', \alpha_1)) \leq \mu(\alpha_1) \times 2^{|\mathcal{P}|}$. Thanks to Proposition 9.12, from (I) and (II) we conclude that $size(I, Keep(I, T, \tilde{\Box}\alpha_1)) \leq (1 + \mu(\alpha_1)) \times 2^{|\mathcal{P}|} = \mu(\tilde{\Box}\alpha_1) \times 2^{|\mathcal{P}|}$.

□

Lemma 12.15. *Let $\alpha \in \mathcal{L}_2$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. For all acceptable sequences N w.r.t. I, if $Keep(I, T, \alpha) \subseteq N$, then for every $t \in N \cap T$, we have $I^N, t \models_{\mathscr{P}} \alpha$.*

Proof. Let $\alpha \in \mathcal{L}_2$ be in NNF, $I = (V, \prec) \in \mathfrak{I}^{sd}$, and let T be a non-empty acceptable sequence w.r.t. I s.t. for all $t \in T$ we have $I, t \models \alpha$. Let N be an acceptable sequence w.r.t. I s.t. $Keep(I, T, \alpha) \subseteq N$ and $t \in N \cap T$ (we assume that N contains at least one $t \in T$). Let $I^N = (N, V^N, \prec^N)$ be the pseudo-interpretation over N.

We use structural induction on T and α in order to prove this property.

- $\alpha = p$ or $\alpha = \neg p$. Since $I, t \models p$ (resp. $\neg p$), it means that $p \in V(t)$ (resp. $p \notin V(t)$). We know that $V^N(t) = V(t)$. We conclude that $I^N, t \models_{\mathscr{P}} p$ (resp. $\neg p$).
- $\alpha = \alpha_1 \wedge \alpha_2$. Since $I, t \models \alpha_1 \wedge \alpha_2$ for all $t \in T$, we can assert that $I, t \models \alpha_1$ and $I, t \models \alpha_2$. By applying the induction hypothesis on T, α_1 and α_2, since $Keep(I, T, \alpha_1) \subseteq N$ and $Keep(I, T, \alpha_2) \subseteq N$, therefore we have $I^N, t \models_{\mathscr{P}} \alpha_1$ and $I^N, t \models_{\mathscr{P}} \alpha_2$. Thus, we have $I^N, t \models_{\mathscr{P}} \alpha_1 \wedge \alpha_2$.
- $\alpha = \alpha_1 \vee \alpha_2$. Suppose that $I, t \models \alpha_1$ (the case $I, t \models \alpha_2$ can be treated in a similar way) and let T_1 be the sequence containing all $t_1 \in T$ s.t. $I, t_1 \models \alpha_1$. Here, since $t \in T_1$, therefore T_1 is non-empty and $t \in T_1 \cap N$. We know that $Keep(I, T_1, \alpha_1) \cup Keep(I, T_2, \alpha_2) \subseteq N$. Consequently $Keep(I, T_1, \alpha_1) \subseteq N$. From the induction hypothesis, we have $I^N, t \models_{\mathscr{P}} \alpha_1$. Therefore, we have $I^N, t \models \alpha_1 \vee \alpha_2$.
- $\alpha = \Diamond\alpha_1$. We have $I, t \models \Diamond\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \Diamond\alpha_1$. $I, t \models \Diamond\alpha_1$ means that there exists $t' \in [t, \infty[$ such that $I, t' \models \alpha_1$. Therefore $Anchors(I, T, \Diamond\alpha_1)$ is non-empty (see Definition 12.9). We know that $Anchors(I, T, \Diamond\alpha_1) \subseteq Keep(I, T, \Diamond\alpha_1) \subseteq N$, consequently $Anchors(I, T, \Diamond\alpha_1) \cap N$ is non-empty. Thanks to Definition 12.9 it is easy to see that for all $t_1 \in Anchors(I, T, \Diamond\alpha_1)$ we have $I, t_1 \models \alpha_1$. By the induction hypothesis on $Anchors(I, T, \Diamond\alpha_1)$ and α_1, since $Keep(I, Anchors(I, T, \Diamond\alpha_1), \alpha_1) \subseteq N$, $t' \in Anchors(I, T, \Diamond\alpha_1)$ (a non-empty acceptable sequence w.r.t I) and $I, t' \models \alpha_1$, thus $I^N, t' \models \alpha_1$. Therefore, we have $I^N, t \models_{\mathscr{P}} \Diamond\alpha_1$.
- $\alpha = \Box\alpha_1$. We have $I, t \models \Box\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \Box\alpha_1$. We know that for all $t' \geq t$ we have $I, t' \models \alpha_1$. We can assert that for all $t' \in N \cap T$ such that $t' \geq t$, we have $I^N, t' \models_{\mathscr{P}} \alpha_1$. By the induction hypothesis on T and α_1, $Keep(I, T, \alpha_1) = Keep(I, T, \Box\alpha_1)$. Consequently $Keep(I, T, \alpha_1) \subseteq N$ since for all $t' \in N \cap T$, we have $I^N, t' \models_{\mathscr{P}} \alpha_1$. We conclude that $I^N, t \models_{\mathscr{P}} \Box\alpha_1$.

- $\alpha = \Diamond\!\!\!\!\sim\alpha_1$. We have $I, t \models \Diamond\!\!\!\!\sim\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \Diamond\!\!\!\!\sim\alpha_1$. $I, t \models \Diamond\!\!\!\!\sim\alpha_1$ means that there exists $t' \in \min_{\prec}(t)$ such that $I, t' \models \alpha_1$, and therefore $\mathit{Anchors}(I, T, \Diamond\!\!\!\!\sim\alpha_1)$ is non-empty (see Definition 12.9). We know that $\mathit{Anchors}(I, T, \Diamond\!\!\!\!\sim\alpha_1) \subseteq \mathit{Keep}(I, T, \Diamond\!\!\!\!\sim\alpha_1) \subseteq N$, consequently $\mathit{Anchors}(I, T, \Diamond\!\!\!\!\sim\alpha_1) \cap N$ is non-empty. Thanks to Definition 12.9 it is easy to see that for all $t_1 \in \mathit{Anchors}(I, T, \Diamond\!\!\!\!\sim\alpha_1)$ we have $I, t_1 \models \alpha_1$. By the induction hypothesis on $\mathit{Anchors}(I, T, \Diamond\!\!\!\!\sim\alpha_1)$ and α_1, since $\mathit{Keep}(I, T_1, \alpha_1) \subseteq N$ with $T_1 = \mathit{Anchors}(I, T, \Diamond\!\!\!\!\sim\alpha_1)$, and T_1 is an acceptable sequence where $I, t' \models \alpha_1$ for all $t' \in T_1$, we conclude (I) $I^N, t' \models_{\mathscr{P}} \alpha_1$. Thanks to the construction of the pseudo-interpretation I^N, since $t' \in \min_{\prec^N}(t)$, we have (II) $t' \in \min_{\prec}(t)$. From (I) and (II), we conclude that $I^N, t \models_{\mathscr{P}} \Diamond\!\!\!\!\sim\alpha_1$.
- $\alpha = \Box\!\!\!\!\sim\alpha_1$. We have $I, t \models \Box\!\!\!\!\sim\alpha_1$ and we need to prove that $I^N, t \models_{\mathscr{P}} \Box\!\!\!\!\sim\alpha_1$. $I, t \models \Box\!\!\!\!\sim\alpha_1$ means that for all $t' \in \min_{\prec}(t)$ we have $I, t' \models \alpha_1$, therefore for all $t' \in T' = \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$ we have $I, t' \models \alpha_1$. In addition, thanks to the well-foundedness condition on $\prec$, T' is non-empty. We know that $\mathit{Anchors}(I, T, \Box\!\!\!\!\sim\alpha_1) \subseteq \mathit{Keep}(I, T, \Box\!\!\!\!\sim\alpha_1) \subseteq N$ and that $\mathit{Anchors}(I, T, \Box\!\!\!\!\sim\alpha_1) = DR(I, T')$, consequently $T' \cap N$ is non-empty. We use proof by contradiction. Suppose that $I^N, t \not\models_{\mathscr{P}} \Box\!\!\!\!\sim\alpha_1$, which means there exists $t' \in \min_{\prec^N}(t_i)$ s.t. $I^N, t' \not\models_{\mathscr{P}} \alpha_1$. Thanks to Proposition 12.12, if $t' \in \min_{\prec^N}(t_i)$, then $t' \in \min_{\prec}(t_i)$. Just a reminder, we have $T' = \bigcup_{t_i \in T} AS(I, \min_{\prec}(t_i))$ where for all $t'' \in T'$ we have $I, t'' \models \alpha_1$. Note that T' is a non-empty acceptable sequence w.r.t. I. By the induction hypothesis on T' and α_1, since $\mathit{Keep}(I, T', \alpha_1) \subseteq N$, and $t' \in AS(I, \min_{\prec}(t)) \subseteq T'$, therefore $I^N, t' \models_{\mathscr{P}} \alpha_1$. This conflicts with our supposition. We conclude that there is no $t' \in \min_{\prec^N}(t)$ s.t. $I^N, t' \not\models_{\mathscr{P}} \alpha_1$, and therefore $I^N, t \models_{\mathscr{P}} \Box\!\!\!\!\sim\alpha_1$.

$\square$

Appendix E. Proof of results in Section 13

NB: The results marked $(*)$ are introduced here, while they are omitted in the main text.

Proposition 13.6. *Let $S = (i, \pi, V_S, \prec_S)$ be a structure, $\mathsf{I}(S) = (V, \prec)$ be its corresponding interpretation and $t, t', t_S, t'_S \in \mathbb{N}$ s.t.:*

$$t_S = \begin{cases} t & \text{if } t < i; \\ i + (t - i) \bmod \pi & \text{otherwise.} \end{cases} \qquad t'_S = \begin{cases} t' & \text{if } t' < i; \\ i + (t' - i) \bmod \pi & \text{otherwise.} \end{cases}$$

We have the following: $t' \in \min_{\prec}(t)$ iff $t'_S \in \min_{\prec_S}(t_S)$.

Proof. Let $S = (i, \pi, V_S, \prec_S)$ be a finite preferential structure, $\mathsf{I}(S) = (V, \prec)$ and $t, t' \in \mathbb{N}$.

- For the only-if part, we use proof by contradiction. We assume that $t' \in \min_{\prec}(t)$ and suppose that $t'_S \notin \min_{\prec_S}(t_S)$. Following the assumption, $t' \in \min_{\prec}(t)$ means that there is no $t'' \in [t, \infty[$ s.t. $(t'', t') \in \prec$. On the other hand, $t'_S \notin \min_{\prec_S}(t_S)$ means that there exists $t''_S \in [\min_{<}\{t_S, i\}, i{+}\pi[$ with $(V_S(t''_S), V_S(t'_S)) \in \prec_S$ (Note that t'_S is also in $[\min_<\{t_S, i\}, i+\pi[)$. Note that following Definition 13.4, we have $V_S(t_S) = V(t_S)$, $V_S(t'_S) = V(t'_S)$ and $V_S(t''_S) = V(t''_S)$. Knowing that $t'_S, t''_S \in [0, i+\pi[$ and $(V_S(t''_S), V_S(t'_S)) \in \prec_S$, then we have $(t''_S, t'_S) \in \prec$. We discuss two cases: $t \in [0, i[$ and $t \in [i, \infty[$.

- ∘ If $t \in [0, i[$, then we have $t = t_S$ and $t''_S \in [t, i+\pi[$ $(t = t_S = [\min_<\{t_S, i\}, i+\pi[)$. Thanks to Definition 13.4, since $t'_S = t'$ in the case of $t' \in [0, i[$ and $t'_S = i + (t' - i) \bmod \pi$ in the case of $t' \in [i, \infty[$, then we have $V(t') = V(t'_S)$. Moreover, since $\mathsf{I}(S) \in \mathfrak{I}^{sd}$, $V(t') = V(t'_S)$ and $(t''_S, t'_S) \in \prec$, then we have $(t''_S, t') \in \prec$. This conflicts with the assumption of $t' \in \min_\prec(t)$.
- ∘ If $t \in [i, \infty[$, then $t_S, t'_S, t''_S \in [i, i+\pi[$ $(t_S \geq i$ and therefore $i = \min_<\{t_S, i\}, i+\pi[$ for both $t'_S, t''_S)$. This entails that $t_S, t'_S, t''_S \in \mathit{final}(\mathsf{I}(S))$. On the hand we have $V(t) = V_S(t_S)$ and $V(t') = V_S(t'_S)$ thanks to Definition 13.4. Thanks to Proposition 9.3, since t''_S and t are in $\mathit{final}(\mathsf{I}(S))$, then there exists $t''' \in [t, \infty$ where $V(t''') = V(t''_S)$. Since $\mathsf{I}(S)$, $V(t''') = V(t''_S)$ and $V(t') = V_S(t'_S)$. This conflicts with the assumption of $t' \in \min_\prec(t)$.

- For the if part, we also use proof by contradiction. We assume that $t'_S \in \min_{\prec_S}(t_S)$ and suppose that $t' \notin \min_\prec(t)$. Following the assumption, $t'_S \in \min_{\prec_S}(t_S)$ entails that there is no $t''_S \in [\min_<\{t_S, i\}, i+\pi[$ with $(V_S(t''_S), V_S(t'_S)) \in \prec_S$. On the other hand, $t' \notin \min_\prec(t)$ means that there exists $t''' \in [t, \infty[$ where $(t'', t') \in \prec$. Let t'''_S be its corresponding points on the finite structure S. t'''_s is defined as follows:

$$t'''_S = \begin{cases} t''' & \text{if } t''' < i; \\ i + (t''' - i) \bmod \pi & \text{otherwise.} \end{cases}$$

The definition of t_S, t'_S, t'''_S in this proposition has two results. The first of which is that t_S, t'_S, t'''_S are all in $[0, i+\pi[$. The second is that, thanks to Definition 13.4, we have $V_S(t_S) = V(t)$, $V_S(t'_S) = V(t')$ and $V_S(t'''_S) = V(t''')$. Moreover, since $(t''', t') \in \prec$, then $(V(t'''), V(t')) \in \prec_S$ and therefore $(V_S(t'''_S), V_S(t'_S)) \in \prec_S$. Next we show that $t'''_S \in [\min_<\{t_S, i\}, i+\pi[$. We discuss two cases.

- ∘ If $t'''_S \in [0, i[$, then we have $t'''_S = t'''$. Moreover, since $t''' \geq t$, we also have $t \in [0, i[$, therefore $t_S = t$ and $t'''_S \in [t_S, i+\pi[$. Thus, we have $t'''_S \in [\min_<\{t_S, i\}, i+\pi[$.
- ∘ The other case is when $t'''_S \in [i, i+\pi[$. It follows that $t'''_S \in [\min_<\{t_S, i\}, i+\pi[$.

Since $(V_S(t'''_S), V_S(t'_S)) \in \prec_S$ and $t'''_S \in [\min_<\{t_S, i\}, i+\pi[$, then $t'_S \notin \min_{\prec_S}(t_S)$. THis conflicts with the assumption of $t'_S \in \min_{\prec_S}(t_S)$.

□

Definition E.1 (∗)**.** Given a UPI $I = (V, \prec)$, we define the finite preferential structure $S(I) = (i, \pi, V_S, \prec_S)$ by:

- $i = \mathit{length}(\mathit{init}(I))$, $\pi = \mathit{card}(\mathit{range}(I))$;
- $V_S(t) = V(t)$ for all $t \in [0, i+\pi[$;
- for all $t, t' \in [0, i+\pi[$, $(V(t), V(t')) \in \prec_S$ iff $(t, t') \in \prec$.

It is worth to note that Definition E.1 is possible because I is an UPI. In particular, UPIs are state-dependent interpretations, i.e., in $\mathfrak{I}^{sd}$. Therefore, for each t, t', t'', t'''

where $V(t) = V(t')$ and $V(t'') = V(t''')$, then $t \prec t''$ iff $t'' \prec t'''$. Thus, it is possible to have a compact representation of the preference relation of UPIs.

Next, we shall show that given an UPI I, the induced interpretation from the finite preferential structure $\mathsf{I}(S(I))$ and I are the same. Let $I = (V, \prec)$ be an UPI and $S(I) = (i, \pi, V_S, \prec_S)$ be its finite preferential structure. Let $\mathsf{I}(S(I)) = (V', \prec')$ be the induced interpretation of $S(I)$. Since $V_S(t) = V(t)$ for all $t \in [0, i+\pi[$ and $V'(t) = V_S(t)$ for all $t \in [0, i+\pi[$, then $V'(t) = V(t)$ for all $t \in [0, i+\pi[$. Given a $t \in [i+\pi, \infty[$, since $V(t) = V(i + (t-i) \bmod \pi)$ (see Definition 13.1, note that $\mathsf{t}_I = \mathit{length}(\mathit{init}(I)) = i$ and $\pi = \mathit{card}(\mathit{range}(I)))$ and $i + (t-i)) \bmod \pi \in [i, i+\pi[$, then $V_S(i + (t-i)) \bmod \pi) = V(i + (t-i) \bmod \pi)$. On the other hand, we have $V'(t) = V_S(i + (t-i)) \bmod \pi)$ for $t \in [i+\pi, \infty[$ (see Definition 13.4), then we have $V'(t) = V(t)$. Therefore, for any $t \in \mathbb{N}$, we have $V'(t) = V(t)$. Moreover, given any $(t, t') \in \prec$, we have $((V(t), V(t')) \in \prec_S)$. Since $(V(t), V(t')) \in \prec_S$, $V(t) = V'(t)$ and $V(t') = V'(t')$, then we have $(t, t') \in \prec'$. The if part follows the same reasoning. Therefore for any $t, t' \in \mathbb{N}$, we have $(t, t') \in \prec$ iff $(t, t') \in \prec'$.

Proposition 13.10. *Let $\alpha \in \mathcal{L}_2$. We have that α is $\mathfrak{I}^{sd}$-satisfiable iff there exists a finite preferential structure S such that $\mathsf{I}(S), 0 \models \alpha$ and $\mathit{size}(\mathsf{I}(S)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.*

Proof. Let $\alpha \in \mathcal{L}_2$.

- For the only if part, let α be $\mathfrak{I}^{sd}$-satisfiable. Thanks to Theorem 12.1 and Proposition 13.2, there exists a UPI $I = (V, \prec) \in \mathfrak{I}^{sd}$ s.t. $I, 0 \models \alpha$ and $\mathit{size}(I) \leq |\alpha| \times 2^{|\mathcal{P}|}$. We define the structure $S(I)$ from I. Since I and $\mathsf{I}(S(I))$ are the same interpretation, then from $\mathfrak{I}^{sd}$-satisfiable sentence α, we can find a finite preferential structure S such that $\mathsf{I}(S), 0 \models \alpha$ and $\mathit{size}(\mathsf{I}(S)) \leq |\alpha| \times 2^{|\mathcal{P}|}$.
- For the if part, let $S = (i, \pi, V_S, \prec_S)$ be a structure s.t. $\mathsf{I}(S), 0 \models \alpha$. Since $\mathsf{I}(S) \in \mathfrak{I}^{sd}$, therefore α is $\mathfrak{I}^{sd}$-satisfiable.

□

Lemma 13.9 is a particular case of the following Lemma.

Lemma E.2 (∗). *Let $S = (i, \pi, V_S, \prec_S)$, $\alpha \in \mathcal{L}_2$ be a finite preferential structure, $\alpha_1 \in \mathit{Sf}(\alpha)$ and $t, t' \in \mathbb{N}$ such that:*

$$t' = \begin{cases} t & \textit{if } t < i; \\ i + (t-i) \bmod \pi & \textit{otherwise.} \end{cases}$$

We have $\mathsf{I}(S), t \models \alpha_1$ iff $\alpha_1 \in \mathit{lab}^S_\alpha(t')$.

Proof. Let $S = (i, \pi, V_S, \prec_S)$ be a finite preferential structure, $\alpha \in \mathcal{L}_2$, $\alpha_1 \in \mathit{Sf}(\alpha)$, $t \in \mathbb{N}$ and $\mathsf{I}(S) = (V, \prec)$. We use structural induction on α_1 to prove the Lemma. Let t' be a time point s.t. $t' = t$, if $t \in [0, i[$, and $t' = i + (t-i) \bmod \pi$, if $t \in [i, \infty[$.

- $\alpha_1 = p$. If $t \in [0, i[$, then we have $V_S(t') = V(t)$. Thus $p \in V_S(t)$ iff $p \in V(t)$, and therefore $\mathsf{I}(S), t \models p$ iff $p \in \mathit{lab}^S_\alpha(t)$. If $t \in [i, \infty[$, we have $V_S(t') = V(t)$. Following the same reasoning as the previous case, $\mathsf{I}(S), t \models p$ iff $p \in \mathit{lab}^S_\alpha(t')$.
- $\alpha_1 = \neg\alpha_2$. By the induction hypothesis, we have $\mathsf{I}(S), t \models \alpha_2$ iff $\alpha_2 \in \mathit{lab}^S_\alpha(t')$, and therefore $\mathsf{I}(S), t \not\models \alpha_2$ iff $\alpha_2 \notin \mathit{lab}^S_\alpha(t')$. We conclude that $\mathsf{I}(S), t \models \neg\alpha_2$ iff $\neg\alpha_2 \in \mathit{lab}^S_\alpha(t')$.

- $\alpha_1 = \alpha_2 \wedge \alpha_3$. By the induction hypothesis, we have $\mathsf{I}(S), t \models \alpha_2$ iff $\alpha_2 \in lab^S_\alpha(t')$ and $\mathsf{I}(S), t \models \alpha_3$ iff $\alpha_3 \in lab^S_\alpha(t')$, and therefore $\mathsf{I}(S), t \models \alpha_2 \wedge \alpha_3$ iff $\alpha_2 \wedge \alpha_3 \in lab^S_\alpha(t')$.
- $\alpha_1 = \Diamond\alpha_2$.
  - For the only-if part, let $\mathsf{I}(S), t \models \Diamond\alpha_2$. We have $t_2 \in [t, \infty[$ s.t. $\mathsf{I}(S), t_2 \models \alpha_2$. Depending on where t_2 is, there exists a t'_2 s.t. $t'_2 = t_2$ if $t_2 \in [0, i[$ and $t'_2 = i + (t_2 - i) \bmod \pi$ if $t_2 \in [i, \infty[$. By the induction hypothesis on α_2, we have $\alpha_2 \in lab^S_\alpha(t'_2)$. Note that t'_2 is $[0, i + \pi[$. Next, we show that $t'_2 \in [\min_<\{t', i\}, i + \pi[$. When $t'_2 \in [0, i[$, then we have $t' = t$ and $t'_2 = t_2$. Since $t \leq t_2$, then $t'_2 \in [t', i[$ and without a loss of generality $t'_2 \in [\min_<\{t', i\}, i+\pi[$. When $t'_2 \in [i, i+\pi[$, it follows that $t'_2 \in [\min_<\{t', i\}, i+\pi[$. In both cases, since $\alpha_2 \in lab^S_\alpha(t'_2)$ and $t'_2 \in [\min_<\{t', i\}, i + \pi[$, we conclude that $\Diamond\alpha_2 \in lab^S_\alpha(t')$ (see Definition 13.8).
  - For the if part, let $\mathsf{I}(S), t \not\models \Diamond\alpha_2$. $\mathsf{I}(S), t \models \neg\Diamond\alpha_2$ means that for all $t_2 \geq t$ we have $\mathsf{I}(S), t_2 \not\models \alpha_2$. By the induction hypothesis, for all $t_2 \geq t$, we have $\alpha_2 \notin lab^S_\alpha(t'_2)$ where $t'_2 = t_2$ if $t_2 \in [0, i[$ and $t'_2 = i + (t_2 - i) \bmod \pi$ if $t_2 \in [i, \infty[$. Following the same reasoning as the only-part proof, we can check that for all $t_2 \geq t$, we have $t'_2 \in [\min_<\{t', i\}, i + \pi[$. Therefore $\alpha_2 \notin lab^S_\alpha(t'_2)$ for all $t'_2 \in [\min_<\{t', i\}, i + \pi[$. Going back to Definition 13.8, we conclude that $\Diamond\alpha_2 \notin lab^S_\alpha(t')$
- $\alpha_1 = \diamondsuit\alpha_2$.
  - For the only-if part, let $\mathsf{I}(S), t \models \diamondsuit\alpha_2$. We have $t_2 \in \min_\prec(t)$ s.t. $\mathsf{I}(S), t_2 \models \alpha_2$. Depending on where t_2 is, there exists a t'_2 s.t. $t'_2 = t_2$ if $t_2 \in [0, i[$ and $t'_2 = i + (t_2 - i) \bmod \pi$ if $t_2 \in [i, \infty[$. By the induction hypothesis on α_2, we have $\alpha_2 \in lab^S_\alpha(t'_2)$. From Proposition 13.6, we can see that $t_2 \in \min_\prec(t)$ iff $t'_2 \in \min_{\prec_S}(t')$. Going back to Definition 13.8, since there is $t'_2 \in \min_{\prec_S}(t')$ where $\alpha_2 \in lab^S_\alpha(t'_2)$, we conclude that $\diamondsuit\alpha_2 \in lab^S_\alpha(t')$.
  - For the if part, let $\mathsf{I}(S), t \not\models \diamondsuit\alpha_2$. $\mathsf{I}(S), t \models \neg\diamondsuit\alpha_2$ means that for all $t_2 \in \min_\prec(t)$ we have $\mathsf{I}(S), t_2 \not\models \alpha_2$. By the induction hypothesis on α_2, for all $t_2 \in \min_\prec(t)$, we have (I) $\alpha_1 \notin lab^S_\alpha(t'_2)$ where $t'_2 = t_2$ if $t_2 \in [0, i[$ and $t'_2 = i + (t_2 - i) \bmod \pi$ if $t_2 \in [i, \infty[$. From Proposition 13.6, we can see that (II) $t_2 \in \min_\prec(t)$ iff $t'_2 \in \min_{\prec_S}(t')$ for all $t_2 \geq t$. Going back to Definition 13.8, since $\alpha_2 \in lab^S_\alpha(t'_2)$ for all $t'_2 \in \min_{\prec_S}(t')$, we conclude that $\diamondsuit\alpha_1 \notin lab^S_\alpha(t')$.

$\square$