Rationality and context in defeasible subsumption

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Abstract. Description logics have been extended in a number of ways to support defeasible reasoning in the KLM tradition. Such features include preferential or rational defeasible concept subsumption, and defeasible roles in complex concept descriptions. Semantically, defeasible subsumption is obtained by means of a preference order on objects, while defeasible roles are obtained by adding a preference order to role interpretations. In this paper, we address an important limitation in defeasible extensions of description logics, namely the restriction in the semantics of defeasible concept subsumption to a single preference order on objects. We do this by inducing a modular preference order on objects from each preference order on roles, and use these to relativise defeasible subsumption. This yields a notion of contextualised rational defeasible subsumption, with contexts described by roles. We also provide a semantic construction for and a method for the computation of contextual rational closure, and present a correspondence result between the two.

1 Introduction

Description Logics (DLs) [2] are decidable fragments of first-order logic that serve as the formal foundation for Semantic-Web ontologies. As witnessed by recent developments in the field, DLs still allow for meaningful, decidable extensions, as new knowledge representation requirements are identified. A case in point is the need to allow for exceptions and defeasibility in reasoning over logic-based ontologies [6, 5, 4, 15, 12, 13, 19, 21, 23–25, 29, 30, 34, 36]. Yet, DLs do not allow for the direct expression of and reasoning with different aspects of defeasibility.

Given the special status of concept subsumption in DLs in particular, and the historical importance of entailment in logic in general, past research efforts in this direction have focused primarily on accounts of defeasible subsumption and the characterisation of defeasible entailment. Semantically, the latter usually takes as point of departure orderings on a class of first-order interpretations, whereas the former usually assume a preference order on objects of the domain.

Recently, we proposed decidable extensions of DLs supporting defeasible knowledge representation and reasoning over ontologies [19, 21, 22]. Our proposal built on previous work to resolve two important ontological limitations of the preferential approach to defeasibility in DLs — the assumption of a single preference order on all objects in the domain of interpretation, and the assumption that defeasibility is intrinsically linked to arguments or conditionals [18, 20].

We achieved this by introducing non-monotonic reasoning features that any classical DL can be extended with in the concept language, in subsumption statements and in role assertions, via an intuitive notion of normality for roles. This parameterised the idea of preference while at the same time introducing the notion of defeasible class membership. Defeasible subsumption allows for the expression of statements of the form "C is usually subsumed by D", for example, "Chenin blanc wines *are usually* unwooded". In the extended language, one can also refer directly to, for example, "Chenin blanc wines that *usually have* a wood aroma". We can also combine these seamlessly, as in: "Chenin blanc wines that *usually have* a wood aroma *are usually* wooded". This cannot be expressed in terms of defeasible subsumption alone, nor can it be expressed w.l.o.g. using typicality-based operators [8, 26, 27] on concepts. This is because the semantics of the expression is inextricably tied to the two distinct uses of the term 'usually'.

Nevertheless, even this generalisation leaves open the question of different, possibly incompatible, notions of defeasibility in subsumption, similar to those studied in contextual argumentation [1, 3]. In the statement "Chenin blanc wines are usually unwooded", the context relative to which the subsumption is normal is left implicit — in this case, the style of the wine. In a different context such as consumer preference or origin, the most preferred (or normal, or typical) Chenin blanc wines may not correlate with the usual wine style. Wine x may be more exceptional than y in one context, but less exceptional in another context. This represents a form of inconsistency in defeasible knowledge bases that could arise from the presence of named individuals in the ontology. The example illustrates why a single ordering on individuals does not suffice. It also points to a natural index for relativised context, namely the use of preferential role names as we have previously proposed [19]. Using role names rather than concept names to indicate context has the advantage that constructs to form complex roles are either absent or limited to role composition.

In this paper, we therefore propose to induce preference orders on objects from preference orders on roles, and use these to relativise defeasible subsumption. This yields a notion of contextualised defeasible subsumption, with contexts described by roles. The remainder of the present paper is structured as follows: in Section 2, we provide a summary of the required background on $\mathcal{ALC}$, the prototypical description logic and on which we shall focus in the present work. In Section 3, we introduce an extension of $\mathcal{ALC}$ to represent both defeasible constructs on complex concepts and contextual defeasible subsumption. In Section 4, we address the most important question from the standpoint of knowledge representation and reasoning with defeasible ontologies, namely that of entailment from defeasible knowledge bases. In particular, we present a semantic construction of contextual rational closure and provide a method for computing it. Finally, with Section 5 we conclude the paper.

We shall assume the reader's familiarity with the preferential approach to non-monotonic reasoning [31, 33, 37]. Whenever necessary, we refer the reader to the definitions and results in the relevant literature.

2 The description logic $\mathcal{ALC}$

The (concept) language of $\mathcal{ALC}$ is built upon a finite set of atomic *concept names* C, a finite set of *role names* R and a finite set of *individual names* I such that C, R and I are pairwise disjoint. With $A, B, \ldots$ we denote atomic concepts, with $r, s, \ldots$ role names, and with $a, b, \ldots$ individual names. Complex concepts are denoted with $C, D, \ldots$ and are built according to the following rule:

$$C ::= \top \mid \bot \mid A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \forall r.C \mid \exists r.C$$

With $\mathcal{L}_{\mathcal{ALC}}$ we denote the *language* of all $\mathcal{ALC}$ concepts.

The semantics of $\mathcal{L}_{\mathcal{ALC}}$ is the standard set theoretic Tarskian semantics. An *interpretation* is a structure $\mathcal{I} := \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, where $\Delta^{\mathcal{I}}$ is a non-empty set called the *domain*, and $\cdot^{\mathcal{I}}$ is an *interpretation function* mapping concept names A to subsets $A^{\mathcal{I}}$ of $\Delta^{\mathcal{I}}$, role names r to binary relations $r^{\mathcal{I}}$ over $\Delta^{\mathcal{I}}$, and individual names a to elements of the domain $\Delta^{\mathcal{I}}$, i.e., $A^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}}$, $r^{\mathcal{I}} \subseteq \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}}$, $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. Define $r^{\mathcal{I}}(x) := \{y \mid (x, y) \in r^{\mathcal{I}}\}$. We extend the interpretation function $\cdot^{\mathcal{I}}$ to interpret complex concepts of $\mathcal{L}_{\mathcal{ALC}}$ in the following way:

$$\top^{\mathcal{I}} := \Delta^{\mathcal{I}}, \quad \bot^{\mathcal{I}} := \emptyset, \quad (\neg C)^{\mathcal{I}} := \Delta^{\mathcal{I}} \setminus C^{\mathcal{I}}$$
$$(C \sqcap D)^{\mathcal{I}} := C^{\mathcal{I}} \cap D^{\mathcal{I}}, \quad (C \sqcup D)^{\mathcal{I}} := C^{\mathcal{I}} \cup D^{\mathcal{I}}$$
$$(\exists r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \cap C^{\mathcal{I}} \neq \emptyset\}, \quad (\forall r.C)^{\mathcal{I}} := \{x \in \Delta^{\mathcal{I}} \mid r^{\mathcal{I}}(x) \subseteq C^{\mathcal{I}}\}$$

Given $C, D \in \mathcal{L}_{\mathcal{ALC}}$, $C \sqsubseteq D$ is called a *subsumption statement*, or *general concept inclusion* (GCI), read "C is subsumed by D". $C \equiv D$ is an abbreviation for both $C \sqsubseteq D$ and $D \sqsubseteq C$. An $\mathcal{ALC}$ *TBox* $\mathcal{T}$ is a finite set of subsumption statements and formalises the *intensional* knowledge about a given domain of application. Given $C \in \mathcal{L}_{\mathcal{ALC}}$, $r \in \mathsf{R}$ and $a, b \in \mathsf{I}$, an *assertional statement* (*assertion*, for short) is an expression of the form $a : C$ or $(a, b) : r$. An $\mathcal{ALC}$ *ABox* $\mathcal{A}$ is a finite set of assertional statements formalising the *extensional* knowledge of the domain. We shall denote statements, both subsumption and assertional, with $\alpha, \beta, \ldots$. Given $\mathcal{T}$ and $\mathcal{A}$, with $\mathcal{KB} := \mathcal{T} \cup \mathcal{A}$ we denote an $\mathcal{ALC}$ *knowledge base*, a.k.a. an *ontology*.

An interpretation $\mathcal{I}$ *satisfies* a subsumption statement $C \sqsubseteq D$ (denoted $\mathcal{I} \Vdash C \sqsubseteq D$) if and only if $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$. $\mathcal{I}$ *satisfies* an assertion $a : C$ (respectively, $(a, b) : r$), denoted $\mathcal{I} \Vdash a : C$ (respectively, $\mathcal{I} \Vdash (a, b) : r$), if and only if $a^{\mathcal{I}} \in C^{\mathcal{I}}$ (respectively, $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in r^{\mathcal{I}}$).

An interpretation $\mathcal{I}$ is a *model* of a knowledge base $\mathcal{KB}$ (denoted $\mathcal{I} \Vdash \mathcal{KB}$) if and only if $\mathcal{I} \Vdash \alpha$ for every $\alpha \in \mathcal{KB}$. A statement α is (classically) *entailed* by $\mathcal{KB}$, denoted $\mathcal{KB} \models \alpha$, if and only if every model of $\mathcal{KB}$ satisfies α.

For more details on Description Logics in general and on $\mathcal{ALC}$ in particular, the reader is invited to consult the Description Logic Handbook [2].

3 Contextual defeasibility in DLs

In this section, we introduce an extension of $\mathcal{ALC}$ to represent both defeasible constructs on complex concepts and contextual defeasible subsumption. The logic presented here draws on the introduction of defeasible roles [19] and recent preliminary work on context-based defeasible subsumption for $\mathcal{SROIQ}$ [21, 22].

3.1 Defeasible constructs

Our previous investigations of defeasible DLs included parameterised defeasible constructs on concepts based on preferential roles, in the form of defeasible value and existential restriction of the form $\tilde{\forall} r.C$ and $\tilde{\exists} r.C$. Intuitively, these concept descriptions refer respectively to individuals whose normal r-relationships are only to individuals from C, and individuals that have some normal r-relationship to an individual from C. However, while these constructs allowed for multiple preference orders on (the interpretation of) roles, only a single preference order on objects was assumed. This was somewhat of an anomaly, which we address here by adding context-based orderings on objects that are derived from preferential roles [21]. Briefly, each preferential role r, interpreted as a strict partial order on the binary product space of the domain, gives rise to a context-based order on objects as detailed in Definition 3 below.

The (concept) language of *defeasible* $\mathcal{ALC}$, or $d\mathcal{ALC}$, is built according to the following rule:

$$C ::= \top \mid \bot \mid A \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \forall r.C \mid \exists r.C \mid \tilde{\forall} r.C \mid \tilde{\exists} r.C$$

With $\mathcal{L}_{d\mathcal{ALC}}$ we denote the language of all $d\mathcal{ALC}$ concepts.

The extension of $\mathcal{ALC}$ we propose here also adds contextual defeasible subsumption statements to knowledge bases. Given $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and $r \in \mathsf{R}$, $C \mathrel{\sqsubset\!\!\!\sim}_r D$ is a *defeasible subsumption statement* or *defeasible GCI*, read "C is usually subsumed by D in the context r". A *defeasible* $d\mathcal{ALC}$ *TBox* $\mathcal{D}$ is a finite set of defeasible GCIs. A *classical* $d\mathcal{ALC}$ *TBox* $\mathcal{T}$ is a finite set of (classical) subsumption statements $C \sqsubseteq D$ (i.e., $\mathcal{T}$ may contain defeasible concept constructs, but not defeasible concept inclusions).

This begs the question of adding some version of "contextual classical subsumption" to the TBox, but, as we shall see in Section 3.2, this simply reduces to classical subsumption.

Given a classical $d\mathcal{ALC}$ TBox $\mathcal{T}$, an ABox $\mathcal{A}$ and a defeasible $d\mathcal{ALC}$ TBox $\mathcal{D}$, from now on we let $\mathcal{KB} := \mathcal{T} \cup \mathcal{D} \cup \mathcal{A}$ and refer to it as a *defeasible* $d\mathcal{ALC}$ *knowledge base* (alias defeasible ontology).

3.2 Preferential semantics

We shall anchor our semantic constructions in the well-known preferential approach to non-monotonic reasoning [31, 33, 37] and its extensions [8, 9, 7, 11, 16–18], especially those in DLs [15, 19, 28, 35].

Let X be a set and let $<$ be a strict partial order on X. With $\min_{<} X := \{x \in X \mid \text{there is no } y \in X \text{ s.t. } y < x\}$ we denote the *minimal elements* of X w.r.t. $<$. With $\#X$ we denote the *cardinality* of X.

Definition 1 (Ordered Interpretation). *An* ***ordered interpretation*** *is a tuple* $\mathcal{O} := \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$ *such that:*

- $\langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}} \rangle$ *is an* $\mathcal{ALC}$ *interpretation, with* $A^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}}$*, for each* $A \in \mathsf{C}$*,* $r^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$*, for each* $r \in \mathsf{R}$*, and* $a^{\mathcal{O}} \in \Delta^{\mathcal{O}}$*, for each* $a \in \mathsf{I}$*, and*
- $\ll^{\mathcal{O}} := \langle \ll_1^{\mathcal{O}}, \ldots, \ll_{\#\mathsf{R}}^{\mathcal{O}} \rangle$*, where* $\ll_i^{\mathcal{O}} \subseteq r_i^{\mathcal{O}} \times r_i^{\mathcal{O}}$*, for* $i = 1, \ldots, \#\mathsf{R}$*, and such that each* $\ll_i^{\mathcal{O}}$ *is a strict partial order and satisfies the smoothness condition [31].*

As an example, suppose $\mathsf{C} := \{A_1, A_2, A_3\}$, $\mathsf{R} := \{r_1, r_2\}$, $\mathsf{I} := \{a_1, a_2, a_3\}$, and $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, with $\Delta^{\mathcal{O}} = \{x_i \mid 1 \leq i \leq 9\}$, $A_1^{\mathcal{O}} = \{x_1, x_4, x_6\}$, $A_2^{\mathcal{O}} = \{x_3, x_5, x_9\}$, $A_3^{\mathcal{O}} = \{x_6, x_7, x_8\}$, $r_1^{\mathcal{O}} = \{(x_1, x_6), (x_4, x_8), (x_2, x_5)\}$, $r_2^{\mathcal{O}} = \{(x_4, x_4), (x_6, x_4), (x_5, x_8), (x_9, x_3)\}$, $a_1^{\mathcal{O}} = x_5$, $a_2^{\mathcal{O}} = x_1$, $a_3^{\mathcal{O}} = x_2$, and $\ll_1^{\mathcal{O}} = \{(x_4x_8, x_2x_5), (x_2x_5, x_1x_6), (x_4x_8, x_1x_6)\}$ and $\ll_2^{\mathcal{O}} = \{(x_6x_4, x_4x_4), (x_5x_8, x_9x_3)\}$. (For the sake of readability, we shall henceforth sometimes write tuples of the form (x, y) as xy.) Figure 1 below depicts the r-ordered interpretation $\mathcal{O}$. In the picture, $\ll_1^{\mathcal{O}}$ and $\ll_2^{\mathcal{O}}$ are represented, respectively, by the dashed and the dotted arrows. (Note the direction of the $\ll^{\mathcal{O}}$-arrows, which point from more preferred to less preferred pairs of objects. Also for the sake of readability, we omit the transitive $\ll^{\mathcal{O}}$-arrows.)

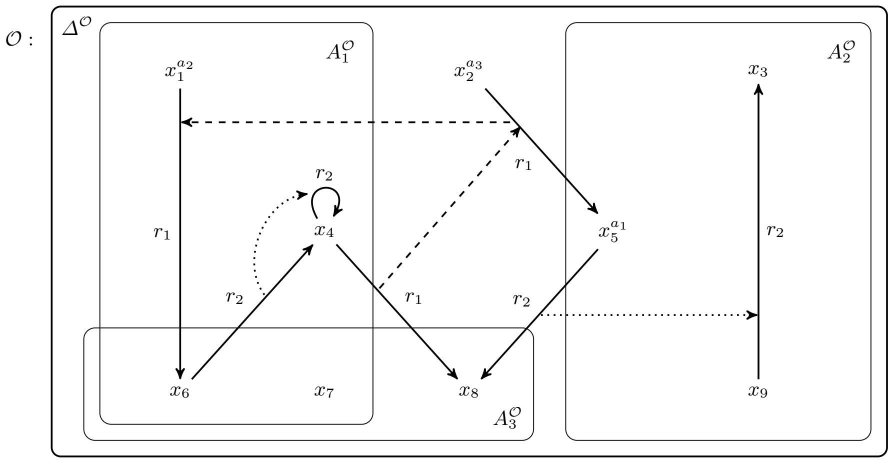

Fig. 1. A $d\mathcal{ALC}$ ordered interpretation.

Given $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, the intuition of $\Delta^{\mathcal{O}}$ and $\cdot^{\mathcal{O}}$ is the same as in a standard DL interpretation. The intuition underlying each of the orderings in $\ll^{\mathcal{O}}$ is that they play the role of *preference relations* (or *normality orderings*), in a sense similar to that introduced by Shoham [37] with a preference on worlds in a propositional setting and as investigated by Kraus et al. [31, 33] and others [11, 14, 26]: the pairs (x, y) that are lower down in the ordering $\ll_i^{\mathcal{O}}$ are deemed as the most normal (or typical, or expected, or conventional) in the context of (the interpretation of) r_i.

In the following definition we show how ordered interpretations can be extended to interpret the complex concepts of the language.

Definition 2 (Concept Interpretation). *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, let $r \in \mathsf{R}$ and let $r_i^{\mathcal{O}|x} := r_i^{\mathcal{O}} \cap (\{x\} \times \Delta^{\mathcal{O}})$ (i.e., the restriction of the domain of $r_i^{\mathcal{O}}$ to $\{x\}$). The interpretation function $\cdot^{\mathcal{O}}$ interprets $d\mathcal{ALC}$ concepts as follows:*

$$\begin{aligned}
&\top^{\mathcal{O}} := \Delta^{\mathcal{O}}; \quad \bot^{\mathcal{O}} := \emptyset;\\
&(\neg C)^{\mathcal{O}} := \Delta^{\mathcal{O}} \setminus C^{\mathcal{O}};\\
&(C \sqcap D)^{\mathcal{O}} := C^{\mathcal{O}} \cap D^{\mathcal{O}};\\
&(C \sqcup D)^{\mathcal{O}} := C^{\mathcal{O}} \cup D^{\mathcal{O}};\\
&(\forall r.C)^{\mathcal{O}} := \{x \mid r^{\mathcal{O}}(x) \subseteq C^{\mathcal{O}}\};\\
&(\forall\!\!\!\sim r.C)^{\mathcal{O}} := \{x \mid \min_{\ll_r^{\mathcal{O}}}(r^{\mathcal{O}|x})(x) \subseteq C^{\mathcal{O}}\};\\
&(\exists r.C)^{\mathcal{O}} := \{x \mid r^{\mathcal{O}}(x) \cap C^{\mathcal{O}} \neq \emptyset\};\\
&(\exists\!\!\!\sim r.C)^{\mathcal{O}} := \{x \mid \min_{\ll_r^{\mathcal{O}}}(r^{\mathcal{O}|x})(x) \cap C^{\mathcal{O}} \neq \emptyset\}.
\end{aligned}$$

If, as in Definition 2, the role name r is not indexed, we use r itself as subscript in $\ll_r^{\mathcal{O}}$. It is not hard to see that, analogously to the classical case, $\forall\!\!\!\sim$ and $\exists\!\!\!\sim$ are dual to each other.

Definition 3 (Satisfaction). *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, $r \in \mathsf{R}$, $C, D \in \mathcal{L}_{d\mathcal{ALC}}$, and $a, b \in \mathsf{I}$. Define $\prec_r^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$ as follows:*

$$\prec_r^{\mathcal{O}} := \{(x, y) \mid (\exists (x, z) \in r^{\mathcal{O}})(\forall (y, v) \in r^{\mathcal{O}})[((x, z), (y, v)) \in \ll_r^{\mathcal{O}}]\}.$$

The satisfaction relation $\Vdash$ is defined as follows:

$\mathcal{O} \Vdash C \sqsubseteq D$ if $C^{\mathcal{O}} \subseteq D^{\mathcal{O}}$;
$\mathcal{O} \Vdash C \sqsubseteq\!\!\!\sim_r D$ if $\min_{\prec_r^{\mathcal{O}}} C^{\mathcal{O}} \subseteq D^{\mathcal{O}}$;
$\mathcal{O} \Vdash a : C$ if $a^{\mathcal{O}} \in C^{\mathcal{O}}$;
$\mathcal{O} \Vdash (a, b) : r$ if $(a^{\mathcal{O}}, b^{\mathcal{O}}) \in r^{\mathcal{O}}$.

If $\mathcal{O} \Vdash \alpha$, then we say $\mathcal{O}$ satisfies α. $\mathcal{O}$ satisfies a defeasible knowledge base $\mathcal{KB}$, written $\mathcal{O} \Vdash \mathcal{KB}$, if $\mathcal{O} \Vdash \alpha$ for every $\alpha \in \mathcal{KB}$, in which case we say $\mathcal{O}$ is a model of $\mathcal{KB}$. We say $C \in \mathcal{L}_{d\mathcal{ALC}}$ is satisfiable w.r.t. $\mathcal{KB}$ if there is a model $\mathcal{O}$ of $\mathcal{KB}$ s.t. $C^{\mathcal{O}} \neq \emptyset$.

It follows from Definition 3 that, if $\ll_r^{\mathcal{O}} = \emptyset$, i.e., if no r-tuple is preferred to another, then $\sqsubseteq\!\!\!\sim_r$ reverts to $\sqsubseteq$. This reflects the intuition that the context r be taken into account through the preference order on $r^{\mathcal{O}}$. In the absence of any preference, the context becomes irrelevant. This also shows why the classical counterpart of $\sqsubseteq\!\!\!\sim_r$ is independent of r — context is taken into account in the form of a preference order, but preference has no bearing on the semantics of $\sqsubseteq$.

The following result, of which the proof extends that in the classical case to deal with preferences, will come in handy in Section 4.2:

Lemma 1. *$d\mathcal{ALC}$ ordered interpretations are closed under disjoint union.*

Lemma 2 below shows that every preferential role interpretation gives rise to a preference order on objects in the domain. Conversely, Lemma 3 shows that every strict partial order on objects in the domain $\Delta^{\mathcal{O}}$ can be obtained from some strict partial order on the interpretation of a new role name as in Definition 3. This means that the more traditional preference order on all objects in the domain is a special case of our proposal.

Lemma 2. *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, $r \in \mathsf{R}$ and let $\prec_r^{\mathcal{O}}$ be as in Definition 3. Then $\prec_r^{\mathcal{O}}$ is a strict partial order on $\Delta^{\mathcal{O}}$.*

Proof. We show that $\prec_r^{\mathcal{O}}$ is (*i*) transitive, (*ii*) irreflexive and (*iii*) antisymmetric.

(*i*) Suppose $(x, y) \in \prec_r^{\mathcal{O}}$ and $(y, z) \in \prec_r^{\mathcal{O}}$. Then $\exists (x, u) \in r^{\mathcal{O}}$ and $\exists (y, v) \in r^{\mathcal{O}}$ such that $(\forall (z, v') \in r^{\mathcal{O}})[((x, u), (y, v)) \in \ll_r^{\mathcal{O}}$ and $((y, v), (z, v')) \in \ll_r^{\mathcal{O}}]$. Since $\ll_r^{\mathcal{O}}$ is transitive, $(x, z) \in \prec_r^{\mathcal{O}}$. Hence $\prec_r$ is transitive.

(*ii*) Suppose $(x, x) \in \prec_r^{\mathcal{O}}$, then $\exists (x, y) \in r^{\mathcal{O}}$ such that $((x, y), (x, y)) \in \ll_r^{\mathcal{O}}$, which contradicts the irreflexivity of $\ll_r^{\mathcal{O}}$. Hence $\prec_r^{\mathcal{O}}$ is irreflexive.

(*iii*) Suppose $(x, y) \in \prec_r^{\mathcal{O}}$ and $(y, x) \in \prec_r^{\mathcal{O}}$. Then $(\exists (x, z) \in r^{\mathcal{O}})(\exists (y, u) \in r^{\mathcal{O}}[((x, z), (y, u)) \in \ll_r^{\mathcal{O}}$ and $((y, u), (x, z)) \in \ll_r^{\mathcal{O}}]$, which contradicts the asymmetry of $\ll$. Hence $\prec_r^{\mathcal{O}}$ is asymmetric (antisymmetric and irreflexive). □

Lemma 3. *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, and let $\prec$ be a strict partial order on $\Delta^{\mathcal{O}}$. Let $\mathcal{O}'$ be an extension of $\mathcal{O}$ with fresh role name $r \in \mathsf{R}$ added, such that $\mathcal{O}' \Vdash \top \sqsubseteq \exists r.\top$, and $\ll_r^{\mathcal{O}'} := \{((x, z), (y, v)) \mid x \prec y$ and $(x, z), (y, v) \in r^{\mathcal{O}'}\}$. Define $\prec_r^{\mathcal{O}'}$ as in Definition 3. Then $\prec = \prec_r^{\mathcal{O}'}$.*

Proof. Suppose $(x, y) \in \prec$. Then x and y are both in the domain of $r^{\mathcal{O}'}$, and $((x, z), (y, v)) \in \ll_r^{\mathcal{O}'}$ for all $(x, z), (y, v) \in r^{\mathcal{O}'}$. Therefore $(x, y) \in \prec_r^{\mathcal{O}'}$. Conversely, suppose $(x, y) \in \prec_r^{\mathcal{O}'}$. Then $(\exists (x, z) \in r^{\mathcal{O}'})(\forall (y, v) \in r^{\mathcal{O}'})[((x, z), (y, v)) \in \ll_r^{\mathcal{O}'}]$. Since y is in the domain of $r^{\mathcal{O}'}$, $(x, y) \in \prec$. □

Corollary 1. *Let $\mathcal{O}'$, $\prec$ and r be as in Lemma 3, and let $\sqsubseteq\!\!\!\sim$ be defined by: $\mathcal{O}' \Vdash C \sqsubseteq\!\!\!\sim D$ if and only if $\min_{\prec} C^{\mathcal{O}'} \subseteq D^{\mathcal{O}'}$. Then $\sqsubseteq\!\!\!\sim_r$ has the same semantics as $\sqsubseteq\!\!\!\sim$.*

Corollary 1 states that, in the special case where the domain of a new designated context-providing role includes all objects, contextual defeasible subsumption coincides with defeasible subsumption based on a single preference order. For the more general parameterised case, consider the role hasOrigin, which links individual wines to origins. Wine y is considered more exceptional than x w.r.t. its origin if it has some more exceptional origin link than x, and none that are less exceptional.

Contextual defeasible subsumption $\sqsubseteq\!\!\!\sim_r$ can therefore also be viewed as defeasible subsumption based on a preference order on objects in the domain of $r^{\mathcal{O}}$, bearing in mind that, in any given interpretation, it is dependent on $\ll_r^{\mathcal{O}}$. For the remainder of this paper, we use $\sqsubseteq\!\!\!\sim$ as abbreviation for $\sqsubseteq\!\!\!\sim_r$, where r is a new role name introduced as in Lemma 3.

This raises the question whether a preference order on objects in the range of $r^{\mathcal{O}}$ could be considered as an alternative. In a more expressive language allowing for role inverses, $\precsim_{\mathsf{inv}(r)}$ achieves this goal [21], but in $d\mathcal{ALC}$, this would have to be added as an additional language construct.

Proposition 1. *For every $r \in \mathsf{R}$, $\sqsubsetsim_r$ is ampliative and non-monotonic:*

- *Ampliativity: for every $\mathcal{O}$, if $\mathcal{O} \Vdash C \sqsubseteq D$, then $\mathcal{O} \Vdash C \sqsubsetsim_r D$;*
- *Non-monotonicity: it is not generally the case that, for every $\mathcal{O}$, if $\mathcal{O} \Vdash C \sqsubsetsim_r D$, then $\mathcal{O} \Vdash C \sqcap E \sqsubsetsim_r D$ for every $E \in \mathcal{L}_{d\mathcal{ALC}}$.*

The following result, of which the proof is analogous to that in the single-ordering case [12], shows that contextual defeasible subsumption is indeed an appropriate notion of non-monotonic subsumption:

Lemma 4. *For every $r \in \mathsf{R}$, $\sqsubsetsim_r$ is a preferential subsumption relation on concepts in that the following rules (a.k.a. KLM-style postulates or properties) hold for every ordered interpretation $\mathcal{O}$, i.e., whenever $\mathcal{O}$ satisfies the rules' antecedent, it satisfies the consequent as well:*

$$\text{(Ref) } C \sqsubsetsim_r C \qquad \text{(LLE) } \frac{C \equiv D,\ C \sqsubsetsim_r E}{D \sqsubsetsim_r E} \qquad \text{(And) } \frac{C \sqsubsetsim_r D,\ C \sqsubsetsim_r E}{C \sqsubsetsim_r D \sqcap E}$$

$$\text{(Or) } \frac{C \sqsubsetsim_r E,\ D \sqsubsetsim_r E}{C \sqcup D \sqsubsetsim_r E} \qquad \text{(RW) } \frac{C \sqsubsetsim_r D,\ D \sqsubseteq E}{C \sqsubsetsim_r E} \qquad \text{(CM) } \frac{C \sqsubsetsim_r D,\ C \sqsubsetsim_r E}{C \sqcap D \sqsubsetsim_r E}$$

We now turn to a class of ordered interpretations that are of special importance in non-monotonic reasoning, namely *modular* interpretations. A strict partial order is called a *modular order* if its set-theoretic complement is a transitive relation.

Definition 4 (Modular Interpretation). *A modular interpretation is an ordered interpretation $\mathcal{O} := \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, where $\ll_r^{\mathcal{O}}$ is modular, for each $r \in \mathsf{R}$.*

We call an ordered model of a knowledge base $\mathcal{KB}$ which is a modular interpretation a *modular model* of $\mathcal{KB}$. It turns out that if the preference order $\ll_r^{\mathcal{O}}$ on the interpretation of r is modular, then the defeasible subsumption $\sqsubsetsim_r$ it induces is also *rational*:

Lemma 5. *For every $r \in \mathsf{R}$, $\sqsubsetsim_r$ is a rational subsumption relation on concepts in that every modular interpretation $\mathcal{O}$ satisfies the following rational monotonicity property:*

$$\text{(RM) } \frac{C \sqsubsetsim_r D,\ C \not\sqsubsetsim_r \neg C'}{C \sqcap C' \sqsubsetsim_r D}.$$

The proof of this lemma is along the lines of that for rationality in the single-ordering case [12] and we do not provide it here.

3.3 Modelling with contexts

The motivation for defeasible knowledge bases is to represent defeasible knowledge, and to reason over defeasible ontologies. We conclude this section with an illustration of the different aspects of defeasibility that can be expressed in $d\mathcal{ALC}$. We first consider defeasible existential restriction:

$$\mathsf{Cheninblanc} \sqcap \tilde{\exists}\, \mathsf{hasAroma.Wood} \sqsubseteq \exists \mathsf{hasStyle.Wooded}$$

This statement is read: “Chenin blanc wines that normally have a wood aroma are wooded”. That is, any Chenin blanc wine that has a characteristic wood aroma, has a wooded wine style. For an example of defeasible subsumption, consider the statement

$$\mathsf{Cheninblanc} \underset{\sim}{\sqsubset} \exists \mathsf{hasAroma.Floral}$$

where $\underset{\sim}{\sqsubset}$ is as in Corollary 1, which states that Chenin blanc wines usually have some floral aroma. That is, the most typical Chenin blanc wines all have some floral aroma. Similarly,

$$\mathsf{Cheninblanc} \underset{\sim}{\sqsubset} \forall \mathsf{hasOrigin.Loire}$$

states that Chenin blanc wines usually come only from the Loire Valley. Now suppose we have a Chenin blanc wine x, which comes from the Loire Valley but does not have a floral aroma, and another Chenin blanc wine y which has a floral aroma but comes from Languedoc. No model of this ontology can simultaneously have $x \prec y$ w.r.t. origin and $y \prec x$ w.r.t. aroma. There can therefore be no model that accurately models reality.

This is precisely the limitation imposed by having only a single ordering on objects, as is broadly assumed by preferential approaches to defeasible DLs [14, 15, 26, 28, 29], and the motivation for introducing context-based defeasible subsumption. Although the two defeasible statements are not inconsistent, the presence of both rules out certain intended models. In contrast, with contextual defeasible subsumption, both subsumption statements can be expressed *and* x and y can have incompatible preferential relationships in the same model:

$$\mathsf{Cheninblanc} \underset{\sim}{\sqsubset}_{\mathsf{hasAroma}} \exists \mathsf{hasAroma.Floral}$$
$$\mathsf{Cheninblanc} \underset{\sim}{\sqsubset}_{\mathsf{hasOrigin}} \forall \mathsf{hasOrigin.Loire}$$

Note that this knowledge base cannot be changed to:

$$\mathsf{Cheninblanc} \sqsubseteq \tilde{\exists} \mathsf{hasAroma.Floral}$$
$$\mathsf{Cheninblanc} \sqsubseteq \tilde{\forall} \mathsf{hasOrigin.Loire}$$

as the latter states that every Chenin blanc wine has a characteristic floral aroma and is usually exclusive to the Loire Valley. This rules out the possibility of a Chenin blanc without a floral aroma, or one that comes only (or just typically) from Languedoc.

We can also add subsumption statements indexed by different contextual roles. For example,

$$\mathsf{Cheninblanc} \sqsubset\!\!\!\sim \exists \mathsf{hasAcidity}.(\mathsf{Medium} \sqcup \mathsf{High})$$
$$\mathsf{Cheninblanc} \sqsubset\!\!\!\sim_{\mathsf{hasOrigin}} \exists \mathsf{hasAcidity}.\mathsf{High}$$

states that Chenin blanc wines usually have a medium or high acidity, whereas Chenin blanc wines of typical origin have a high acidity.

4 Entailment in $d\mathcal{ALC}$

Given a defeasible $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$, we are interested in the reasoning task of *entailment of statements* from $\mathcal{KB}$. That is, given the knowledge specified in $\mathcal{KB}$, how do we decide what other subsumption statements follow from $\mathcal{KB}$? In Section 4.1, we first introduce the natural generalisation of entailment to a preferential setting. Thereafter we consider the additional assumption of modularity on preferential models. This serves as motivation for our semantic characterisation of rational entailment in Section 4.2.

4.1 Preferential entailment

In order to get to a definition of entailment for $d\mathcal{ALC}$, an obvious starting point is to adopt a Tarskian notion thereof:

Definition 5 (Preferential Entailment). *A statement α is preferentially entailed by a defeasible $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\mathsf{pref}} \alpha$, if every ordered model of $\mathcal{KB}$ satisfies α.*

When assessing how appropriate a notion of entailment is, a task we shall devote time to in this section, the following definitions come in handy, as it will become clear in the sequel:

Definition 6. *A defeasible $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$ is called preferential if it is closed under the preferential rules in Lemma 4.*

Definition 7 (Preferential Closure). *Let $\mathcal{KB}$ be a defeasible $d\mathcal{ALC}$ knowledge base. With*

$$\mathcal{KB}^*_{\mathsf{pref}} := \bigcap \{\mathcal{KB}' \mid \mathcal{KB} \subseteq \mathcal{KB}' \text{ and } \mathcal{KB}' \text{is preferential}\}$$

we denote the preferential closure of $\mathcal{KB}$.

Intuitively, the preferential closure of a defeasible $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$ corresponds to the 'core' set of statements that hold given those in $\mathcal{KB}$. It provides an alternative, and, in our context, quite convenient, way to look at entailment, as the following result shows:

Lemma 6. *Let $\mathcal{KB}$ be a defeasible $d\mathcal{ALC}$ knowledge base and let α be a statement. Then $\mathcal{KB} \models_{\mathsf{pref}} \alpha$ iff $\alpha \in \mathcal{KB}^*_{\mathsf{pref}}$.*

Hence, preferential entailment and preferential closure are two sides of the same coin, mimicking an analogous result for preferential reasoning in both the propositional [31] and the DL [12, 15] cases. A further feature of preferential closure (and, therefore, of preferential entailment) is the following:

Lemma 7. *$\mathcal{KB}^*_{\mathsf{pref}}$ is preferential.*

In other words, preferential entailment ensures that the set of statements (in particular the $\sqsubsetsim_r$-ones) that follow from the knowledge base satisfies the $d\mathcal{ALC}$ versions of the basic KLM-style properties for defeasible reasoning (cf. Lemma 4).

Of course, preferential entailment is not always desirable, one of the reasons being that it is monotonic, courtesy of the Tarskian notion of consequence it relies on (see Definition 5). In most cases, as witnessed by the great deal of work in the non-monotonic reasoning community, a move towards rationality is in order. Thanks to the definitions above and the result in Lemma 5, we already know where to start looking for it:

Definition 8 (Modular Entailment). *A statement α is modularly entailed by a defeasible $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$, written $\mathcal{KB} \models_{\mathsf{mod}} \alpha$, if every modular model of $\mathcal{KB}$ satisfies α.*

We say a defeasible $d\mathcal{ALC}$ knowledge base $\mathcal{KB}$ is rational if it is closed under the preferential rules in Lemma 4 and the rational mononotonicity rule in Lemma 5.

Definition 9 (Modular Closure). *Let $\mathcal{KB}$ be a defeasible $d\mathcal{ALC}$ knowledge base. With*

$$\mathcal{KB}^*_{\mathsf{mod}} := \bigcap\{\mathcal{KB}' \mid \mathcal{KB} \subseteq \mathcal{KB}' \text{ and } \mathcal{KB}' \text{is rational}\}$$

we denote the modular closure of $\mathcal{KB}$.

Just as in the preferential case, it turns out modular closure and modular entailment coincide:

Lemma 8. *Let $\mathcal{KB}$ be a defeasible $d\mathcal{ALC}$ knowledge base and let α be a statement. Then $\mathcal{KB} \models_{\mathsf{mod}} \alpha$ iff $\alpha \in \mathcal{KB}^*_{\mathsf{mod}}$.*

Unfortunately, modular closure (and modular entailment) falls short of providing us with an appropriate notion of non-monotonic entailment. This is so because it coincides with preferential closure, as the following result, adapted from a well-known similar result in the propositional case [33, Theorem 4.2], shows.

Lemma 9. *$\mathcal{KB}^*_{\mathsf{mod}} = \mathcal{KB}^*_{\mathsf{pref}}$.*

More fundamentally, this means the set of $\sqsubsetsim$-statements that are modularly entailed by a knowledge base need not satisfy the rational monotonicity property, since $\mathcal{KB}^*_{\mathsf{mod}}$ (or $\mathcal{KB}^*_{\mathsf{pref}}$) is not, in general, rational. In what follows, we overcome precisely this issue.

4.2 Rational entailment

In this section, we introduce a definition of semantic entailment which, as we shall see, is appropriate in the light of the discussion above. The constructions we are going to present are inspired by the work by Booth and Paris [10] in the propositional case and those by Britz et al. [12] and Giordano et al. [29, 30] in a single-ordered preferential DL setting. (We shall give a corresponding proof-theoretic characterisation of such a notion of entailment in Section 4.3.)

Let $\mathcal{KB}$ be a defeasible knowledge base and let Δ be a fixed countably infinite set. We define $\mathcal{O}_{\Delta}^{\mathcal{KB}} := \{\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle \mid \mathcal{O} \Vdash \mathcal{KB}$ and $\mathcal{O}$ is modular and $\Delta^{\mathcal{O}} = \Delta\}$. The following result shows that the set $\mathcal{O}_{\Delta}^{\mathcal{KB}}$ characterises modular entailment:

Lemma 10. *For every $\mathcal{KB}$, every $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $r \in \mathsf{R}$, $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq\!\!\!\!\sim_r D$ iff $\mathcal{O} \Vdash C \sqsubseteq\!\!\!\!\sim_r D$, for every $\mathcal{O} \in \mathcal{O}_{\Delta}^{\mathcal{KB}}$.*

Since Δ is countable, for every $\mathcal{O} \in \mathcal{O}_{\Delta}^{\mathcal{KB}}$, we can partition $\Delta \times \Delta$ into a sequence of layers $(L_0, \ldots, L_n, \ldots)$, where, for each $i \geq 0$, $L_i := \langle L_i^{r_1}, \ldots, L_i^{r_{\#\mathsf{R}}} \rangle$, and such that, for every $x, y \in \Delta$ and every $r \in \mathsf{R}$, $(x, y) \in L_0^r$ iff $(x, y) \in \min_{\ll_r} r^{\mathcal{O}}$ and $(x, y) \in L_{i+1}^r$ iff $(x, y) \in \min_{\ll_r}(r^{\mathcal{O}} \setminus \bigcup_{0 \leq j \leq i} L_j^r)$. (That these constructions are well defined follows from the fact that for every $r \in \mathsf{R}$, $\ll_r$ is smooth.)

Definition 10 (Height of a pair). *Let $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle$, let $x, y \in \Delta^{\mathcal{O}}$ and let $r \in \mathsf{R}$. The height of (x, y) in $\mathcal{O}$ w.r.t. r is denoted $h_{\mathcal{O}}(x, y, r)$ and is equal to i iff $(x, y) \in L_i^r$.*

We can now use the set $\mathcal{O}_{\Delta}^{\mathcal{KB}}$ as a springboard to introduce a version of 'canonical' modular interpretation.

Definition 11 (Big modular interpretation). *Let $\mathcal{KB}$ be a defeasible knowledge base and define $\mathcal{O}_{\oplus}^{\mathcal{KB}} := \langle \Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}, \cdot^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}, \ll^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} \rangle$, where*

- *$\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} := \bigoplus_{\mathcal{O} \in \mathcal{O}_{\Delta}^{\mathcal{KB}}} \Delta^{\mathcal{O}}$, i.e., the disjoint union of the domains from $\mathcal{O}_{\Delta}^{\mathcal{KB}}$, where each $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, \ll^{\mathcal{O}} \rangle \in \mathcal{O}_{\Delta}^{\mathcal{KB}}$ has the elements $x, y, \ldots$ of its domain renamed as $x_{\mathcal{O}}$, $y_{\mathcal{O}}$, $\ldots$ so that they are all distinct in $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$;*
- *$x_{\mathcal{O}} \in A^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ iff $x \in A^{\mathcal{O}}$;*
- *$(x_{\mathcal{O}}, y_{\mathcal{O}'}) \in r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ iff $\mathcal{O} = \mathcal{O}'$ and $(x, y) \in r^{\mathcal{O}}$;*
- *$(x_{\mathcal{O}}, y_{\mathcal{O}'}) \ll_r^{\mathcal{O}_{\oplus}^{\mathcal{KB}}} (x'_{\mathcal{O}}, y'_{\mathcal{O}'})$ iff $h_{\mathcal{O}}(x, y, r) < h_{\mathcal{O}'}(x', y', r)$.*

The proofs for the two lemmas below follow from the definition of $\mathcal{O}_{\oplus}^{\mathcal{KB}}$:

Lemma 11. *For every $C \in \mathcal{L}_{d\mathcal{ALC}}$, $x_{\mathcal{O}} \in C^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ iff $x \in C^{\mathcal{O}}$.*

Lemma 12. *For every $r \in \mathsf{R}$, $h_{\mathcal{O}_{\oplus}^{\mathcal{KB}}}(x_{\mathcal{O}}, y_{\mathcal{O}}, r) = h_{\mathcal{O}}(x, y, r)$.*

These results, together with the fact that $d\mathcal{ALC}$ modular interpretations are closed under disjoint union (Lemma 1), allow us to show the following:

Lemma 13. $\mathcal{O}_{\oplus}^{\mathcal{KB}}$ *is a modular model of* $\mathcal{KB}$.

Given $\mathcal{O}_{\oplus}^{\mathcal{KB}}$, we can then define contextual modular orderings on the domain $\Delta^{\mathcal{O}_{\oplus}^{\mathcal{KB}}}$ in the same way as in Definition 3.

Armed with the definitions and results above, we are now ready to provide an alternative definition of entailment in $d\mathcal{ALC}$:

Definition 12 (Rational Entailment). *A statement* α *is rationally entailed by a knowledge base* $\mathcal{KB}$*, written* $\mathcal{KB} \models_{\mathsf{rat}} \alpha$*, if* $\mathcal{O}_{\oplus}^{\mathcal{KB}} \Vdash \alpha$.

That such a notion of entailment indeed deserves its name is witnessed by the following result:

Lemma 14. *Let* $\mathcal{KB}$ *be a defeasible knowledge base. For every* $r \in \mathsf{R}$, $\{C \sqsubsim_r D \mid \mathcal{O}_{\oplus}^{\mathcal{KB}} \Vdash C \sqsubsim_r D\}$ *is rational.*

4.3 Computing contextual rational closure

In the remaining of the section, we discuss a known instance of entailment for defeasible reasoning that meets all the requirements of rational entailment. It is a generalisation of the DL version of the propositional *rational closure* studied by Lehmann and Magidor [33], to deal with context-based rational defeasible entailment. We present a proof-theoretic characterisation here, based on the work of Casini and Straccia [24, 25]; an alternative semantic characterisation of rational closure in DLs (without contexts) was proposed by Giordano and others [29, 30].

Rational closure is a form of inferential closure based on modular entailment $\models_{\mathsf{mod}}$, but it extends its inferential power. Such an extension of modular entailment is obtained formalising what is called the *presumption of typicality* [32, Section 3.1]. That is, we always assume that we are dealing with the most typical possible situation compatible with the information at our disposal. We first define what it means for a concept to be *exceptional* in a given context:

Definition 13 (Contextual Exceptionality). *A concept* C *is exceptional in the context* r *in the defeasible knowledge base* $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ *if* $\mathcal{KB} \models_{\mathsf{mod}} \top \sqsubsim_r \neg C$. *A defeasible subsumption statement* $C \sqsubsim_r D$ *is exceptional in the context* r *in* $\mathcal{KB}$ *if* C *is exceptional in the context* r *in* $\mathcal{KB}$.

So, a concept C is considered exceptional in a given context in a knowledge base if it is not possible to have a modular model of the knowledge base in which there is a typical individual (i.e., an individual at least as typical as all the others) that is an instance of the concept C. Applying the notion of exceptionality iteratively, we associate with every concept C and context r a *rank* in the knowledge base $\mathcal{KB}$, which we denote by $\mathsf{rank}_{\mathcal{KB}}(C, r)$. We extend this to subsumption statements, and associate with every context r and contextual defeasible concept inclusion $C \sqsubsim_r D$ a rank, denoted $\mathsf{rank}_{\mathcal{KB}}(C \sqsubsim_r D, r)$ and abbreviated as $\mathsf{rank}_{\mathcal{KB}}(C \sqsubsim_r D)$:

1. Let $\mathsf{rank}_{\mathcal{KB}}(C, r) = 0$ if C is not exceptional in the context of r and $\mathcal{KB}$, and let $\mathsf{rank}_{\mathcal{KB}}(C \mathrel{\sqsubseteq\!\!\!\sim_r} D) = 0$ for every defeasible statement having C as antecedent, with $\mathsf{rank}_{\mathcal{KB}}(C, r) = 0$. The set of statements in $\mathcal{D}$ with rank 0 is denoted as $\mathcal{D}_0^{\mathsf{rank}}$.
2. Let $\mathsf{rank}_{\mathcal{KB}}(C, r) = 1$ if C does not have a rank of 0 in the context of r and it is not exceptional in the knowledge base $\mathcal{KB}^1$ composed of $\mathcal{T}$ and the exceptional part of $\mathcal{D}$, that is, $\mathcal{KB}^1 = \langle \mathcal{T}, \mathcal{D} \setminus \mathcal{D}_0^{\mathsf{rank}} \rangle$. If $\mathsf{rank}_{\mathcal{KB}}(C, r) = 1$, then let $\mathsf{rank}_{\mathcal{KB}}(C \mathrel{\sqsubseteq\!\!\!\sim_r} D) = 1$ for every statement $C \mathrel{\sqsubseteq\!\!\!\sim_r} D$. The set of statements in $\mathcal{D}$ with rank 1 is denoted $\mathcal{D}_1^{\mathsf{rank}}$.
3. In general, for $i > 0$, a tuple $\langle C, r \rangle$ is assigned a rank of i if it does not have a rank of $i - 1$ and it is not exceptional in $\mathcal{KB}^i = \langle \mathcal{T}, \mathcal{D} \setminus \bigcup_{j=0}^{i-1} \mathcal{D}_j^{\mathsf{rank}} \rangle$. If $\mathsf{rank}_{\mathcal{KB}}(C, r) = i$, then $\mathsf{rank}_{\mathcal{KB}}(C \mathrel{\sqsubseteq\!\!\!\sim_r} D) = i$ for every statement $C \mathrel{\sqsubseteq\!\!\!\sim_r} D$. The set of statements in $\mathcal{D}$ with rank i is denoted $\mathcal{D}_i^{\mathsf{rank}}$.
4. By iterating the previous steps, we eventually reach a subset $\mathcal{E} \subseteq \mathcal{D}$ such that all the statements in $\mathcal{E}$ are exceptional (since $\mathcal{D}$ is finite, we must reach such a point). If $\mathcal{E} \neq \emptyset$, we define the rank of the statements in $\mathcal{E}$ as ∞, and the set $\mathcal{E}$ is denoted $\mathcal{D}_\infty^{\mathsf{rank}}$.

Following on the procedure above, $\mathcal{D}$ is partitioned into a finite sequence $\langle \mathcal{D}_0^{\mathsf{rank}}, \ldots, \mathcal{D}_n^{\mathsf{rank}}, \mathcal{D}_\infty^{\mathsf{rank}} \rangle$ $(n \geq 0)$, where $\mathcal{D}_\infty^{\mathsf{rank}}$ may possibly be empty. So, through this procedure we can assign a rank to every context-based defeasible subsumption statement.

For a concept C to have a rank of ∞ corresponds to not being satisfiable in any model of $\mathcal{KB}$, that is, $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq \bot$. Note that this relationship is independent of context:

Lemma 15. *Let $C \in \mathcal{L}_{d\mathcal{ALC}}$. Then $\mathsf{rank}_{\mathcal{KB}}(C, r) = \infty$ for all $r \in \mathsf{R}$ if and only iff $\mathcal{KB} \models_{\mathsf{mod}} C \sqsubseteq \bot$.*

Adapting Lehmann and Magidor's construction for propositional logic [33], the contextual rational closure of a knowledge base $\mathcal{KB}$ is defined as follows:

Definition 14 (Contextual Rational Closure). *Let $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and let $r \in \mathsf{R}$. Then $C \mathrel{\sqsubseteq\!\!\!\sim_r} D$ is in the rational closure of a defeasible knowledge base $\mathcal{KB}$ if*

$$\mathsf{rank}_{\mathcal{KB}}(C \sqcap D, r) < \mathsf{rank}_{\mathcal{KB}}(C \sqcap \neg D, r) \text{ or } \mathsf{rank}_{\mathcal{KB}}(C) = \infty \ .$$

Informally, the above definition says that $C \mathrel{\sqsubseteq\!\!\!\sim_r} D$ is in the rational closure of $\mathcal{KB}$ if the ranked models of the knowledge base tell us that, in the context of r, some instances of $C \sqcap D$ are more plausible than all instances of $C \sqcap \neg D$.

Theorem 1. *Let $\mathcal{KB}$ be a knowledge base having a modular model. For every $C, D \in \mathcal{L}_{d\mathcal{ALC}}$ and every $r \in \mathsf{R}$, $C \mathrel{\sqsubseteq\!\!\!\sim_r} D$ is in the rational closure of $\mathcal{KB}$ iff $\mathcal{KB} \models_{\mathsf{rat}} C \mathrel{\sqsubseteq\!\!\!\sim_r} D$.*

4.4 Rational reasoning with contextual ontologies

The following example shows how ranks are assigned to concepts in a defeasible TBox, and used to determine rational entailment. We first consider only a single context $\mathsf{hasE} \in \mathsf{R}$ with intuition 'has employment', and then extend the example to demonstrate the strength of reasoning with multiple contexts.

Let $\mathcal{KB} = \mathcal{T} \cup \mathcal{D}$ with $\mathcal{T} = \{\mathsf{Intern} \sqsubseteq \mathsf{Employee},\ \mathsf{Employee} \sqsubseteq \exists \mathsf{hasE}.\top\}$ and

$$\mathcal{D} = \left\{ \begin{array}{c} \mathsf{Employee} \sqsubsetsim_{\mathsf{hasE}} \exists \mathsf{hasID}.\mathsf{TaxNo}, \\ \mathsf{Intern} \sqsubsetsim_{\mathsf{hasE}} \neg\exists \mathsf{hasID}.\mathsf{TaxNo}, \\ \mathsf{Intern} \sqcap \mathsf{Graduate} \sqsubsetsim_{\mathsf{hasE}} \exists \mathsf{hasID}.\mathsf{TaxNo} \end{array} \right\}$$

Examining the concepts on the LHS of each subsumption in $\mathcal{D}$, we get that:

1. $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Employee}, \mathsf{hasE}) = 0$, since $\mathsf{Employee}$ is not exceptional in $\mathcal{KB}$.
2. $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern}, \mathsf{hasE}) \neq 0$ and $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) \neq 0$, since both concepts are exceptional in $\mathcal{KB}$.
3. $\mathcal{KB}^1$ is composed of $\mathcal{T}$ and $\mathcal{D} \setminus \mathcal{D}_0^{\mathsf{rank}}$, which consists of the defeasible subsumptions in $\mathcal{D}$ except for $\mathsf{Employee} \sqsubsetsim_{\mathsf{hasE}} \exists \mathsf{hasID}.\mathsf{TaxNo}$.
4. $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern}, \mathsf{hasE}) = 1$, since Intern is not exceptional in $\mathcal{KB}^1$.
5. $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) \neq 1$, since $\mathsf{Intern} \sqcap \mathsf{Graduate}$ is exceptional in $\mathcal{KB}^1$.
6. $\mathcal{KB}^2$ is composed of $\mathcal{T}$ and $\{\mathsf{Intern} \sqcap \mathsf{Graduate} \sqsubsetsim_{\mathsf{hasE}} \exists \mathsf{hasID}.\mathsf{TaxNo}\}$.
7. $\mathsf{Intern} \sqcap \mathsf{Graduate}$ is not exceptional in $\mathcal{KB}^2$ and therefore $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) = 2$.

There are algorithms to compute rational closure [23, 25, 30] that can readily be adapted to account for context, but one can also apply Definition 14 to determine rational entailment. For example, since $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) = 2$ but $\mathsf{rank}_{\mathcal{KB}}(\mathsf{Intern} \sqcap \neg\mathsf{Graduate}, \mathsf{hasE}) = 1$, we find that interns are usually not graduates: $\mathcal{KB} \models_{\mathsf{rat}} \mathsf{Intern} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$.

The context hasE is used to indicate that it is an individual's typicality in the context of employment which is under consideration. Now suppose that $\mathcal{KB}$ in the above example is extended to $\mathcal{KB}' = \langle \mathcal{T}', \mathcal{D}' \rangle$, where $\mathcal{T}' = \mathcal{T}$ and $\mathcal{D}' = \mathcal{D} \cup \{\mathsf{Millennial} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Employee},\ \mathsf{Millennial} \sqsubsetsim_{\mathsf{hasQ}} \mathsf{Graduate}\}$. The context hasQ is used here to indicate that it is an individual's typicality w.r.t. qualification which is under consideration. The rankings calculated above remain unchanged; in addition, we get $\mathsf{rank}_{\mathcal{KB}'}(\mathsf{Millennial}, \mathsf{hasE}) = 0$ and $\mathsf{rank}_{\mathcal{KB}'}(\mathsf{Millennial}, \mathsf{hasQ}) = 0$. It now follows that:

- In the context hasQ, millennial interns are usually graduates: $\mathcal{KB}' \models_{\mathsf{rat}} \mathsf{Millennial} \sqcap \mathsf{Intern} \sqsubsetsim_{\mathsf{hasQ}} \mathsf{Graduate}$. This follows because $\mathsf{rank}_{\mathcal{KB}'}(\mathsf{Millennial} \sqcap \mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasQ}) = 0$, whereas $\mathsf{rank}_{\mathcal{KB}'}(\mathsf{Millennial} \sqcap \mathsf{Intern} \sqcap \neg\mathsf{Graduate}, \mathsf{hasQ}) = 1$.
- In the context hasE, millennial interns are usually not graduates: $\mathcal{KB}' \models_{\mathsf{rat}} \mathsf{Millennial} \sqcap \mathsf{Intern} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$. This follows because $\mathsf{rank}_{\mathcal{KB}'}(\mathsf{Millennial} \sqcap \mathsf{Intern} \sqcap \mathsf{Graduate}, \mathsf{hasE}) = 2$, whereas $\mathsf{rank}_{\mathcal{KB}'}(\mathsf{Millennial} \sqcap \mathsf{Intern} \sqcap \neg\mathsf{Graduate}, \mathsf{hasE}) = 1$.

On the other hand, suppose we were restricted to a single context hasE, i.e., replace hasQ with hasE in $\mathcal{KB}'$ to obtain $\mathcal{KB}''$. We then only get that $\mathcal{KB}'' \models_{\mathsf{rat}} \mathsf{Millennial} \sqcap \mathsf{Intern} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$.

Which one of these rational entailments is more intuitively correct depends (*sic*) on the context, and can perhaps be understood better by looking at the postulates for non-monotonic reasoning in Lemmas 4 and 5. Looking at models of $\mathcal{KB}'$, in particular $\mathcal{O}_{\oplus}^{\mathcal{KB}'}$, it follows from (RM) that $\mathcal{KB}' \models_{\mathsf{rat}} \mathsf{Millennial} \sqcap \mathsf{Intern} \sqsubsetsim_{\mathsf{hasQ}} \mathsf{Graduate}$. That is, in the context of qualifications, since millennials are usually graduates, so are millennial interns. Also in $\mathcal{KB}'$, applying (RM) to $\mathsf{Intern} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$ we get $\mathsf{Intern} \sqcap \mathsf{Millennial} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Graduate}$. That is, in the context of employment, since interns are usually not graduates, neither are millennial interns.

In contrast, in models of $\mathcal{KB}''$, including the big model $\mathcal{O}_{\oplus}^{\mathcal{KB}''}$, the former deduction is blocked: Applying (RW) to $\mathsf{Millennial} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Employee}$ yields $\mathsf{Millennial} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Intern}$. (RM) is now blocked by $\mathsf{Millennial} \sqsubsetsim_{\mathsf{hasE}} \neg\mathsf{Intern}$, hence we cannot conclude that $\mathcal{KB}'' \models_{\mathsf{rat}} \mathsf{Millennial} \sqcap \mathsf{Intern} \sqsubsetsim_{\mathsf{hasE}} \mathsf{Graduate}$.

5 Concluding remarks

In this paper, we have made a case for a context-based notion of defeasible concept inclusion in description logics. We have shown that preferential roles can be used to take context into account, and to deliver a simple, yet powerful, notion of contextual defeasible subsumption. Technically, this addresses an important limitation in previous defeasible extensions of description logics, namely the restriction in the semantics of defeasible concept inclusion to a single preference order on objects. Semantically, it answers the question of the meaning of multiple preference orders, namely that they reflect different contexts.

Building on previous work in the KLM tradition, we have shown that restricting the preferential semantics to a modular semantics allows us to define the notion of rational entailment from a defeasible knowledge base, and to compute the rational closure of a knowledge base as an instance of rational entailment. Future work should consider the implementation of contextual rational closure, as well as the addition of an ABox. Much work is also required on the modelling side once a stable implementation exists. Contextual subsumption provides the user with more flexibility in making defeasible statements in ontologies, but comprehensive case studies are required to evaluate the approach.

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