Preferential Modalities Revisited

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Abstract

We venture beyond the customary semantic approach in NMR, namely that of placing orderings on worlds (or valuations). In a modal-logic setting, we motivate and investigate the idea of ordering elements of the accessibility relations in Kripke frames, i.e., world pairs (w, w') (or 'arrows'). The underlying intuition is that some world pairs may be seen as more normal (or typical, or expected) than others. We show this delivers an elegant and intuitive semantic construction, which gives a new perspective on present notions of defeasible necessity. From a modeler's perspective, the new framework we propose is more intuitively appealing. Technically, though, the revisited logic happens to not substantively increase the expressive power of the previously defined preferential modalities. This conclusion follows from an analysis of both semantic constructions via a generalisation of bisimulations to the preferential case. Lest this be seen as a negative result, it essentially means that reasoners based on the previous semantics (which have been shown to preserve the computational complexity of the underlying classical modal language) suffice for reasoning over the new semantics. Finally, we show that the kind of construction we here propose has many fruitful applications, notably in a description-logic context, where it provides the foundations on which to ground useful notions of defeasibility in ontologies yet to be explored.

Introduction

Accounts of normality (or typicality), plausibility and alike traditionally have an underlying semantics built on a notion of preference on *worlds*. Such is the case of non-monotonic entailment (Shoham 1988; Kraus, Lehmann, and Magidor 1990; Makinson 2005), conditionals (Lehmann and Magidor 1992; Boutilier 1994), belief revision (Katsuno and Mendelzon 1991; Baltag and Smets 2006; 2008), counterfactuals (Stalnaker 1968; Lewis 1973; 1974), obligations (Hansson 1969) and many others, as known from the literature on non-monotonic reasoning, conditional and deontic logics, and related areas. Roughly speaking, the usual approach consists in selecting some worlds (or propositional valuations) as being more normal (alias typical, alias desirable) and carrying out the reasoning relative to an underlying normality ordering on worlds.

A typical representative of these different yet interrelated threads of investigation is the well-known preferential approach (Shoham 1988) and its derivatives (Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992). There, a preference relation is defined on the set of possible worlds with the (tacit) assumption that these contain all is needed to reason about what is normal or expected. A case can indeed be made for such an assumption in a propositional setting. However, in logics with more structure, it is reasonable to say that the normality 'spotlight' should not be confined to worlds, but rather (also) be put on (possibly) whatever structure one has at one's disposal in the respective underlying semantics. To witness, in a modal logic context, it makes sense to ask whether some links between worlds in a frame are (relatively) more normal (or preferred) than others—irrespective of whether the worlds involved are by any means comparable in that way or another amongst themselves. In other words, one can be interested in the normality of the *transition* from one world to another one. This point is better illustrated with some well-known concrete applications of modal logics as given below.

Let us assume a very simplistic scenario in which we have only one propositional atom, on, of which the intuition is that a particular light-bulb is on. Moreover, let us assume there is only one action at one's disposal, namely toggle (hereafter abbreviated t), of which the intuition is that of changing the state of the light switch. Figure 1 below depicts a possible-worlds model for this scenario.

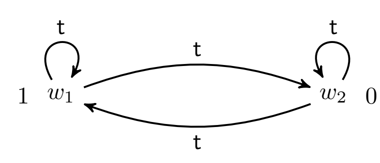

Figure 1: A possible-worlds model for one action (toggle) and one atom (on).

Intuitively, normal (or typical, or expected) executions of the toggle action are given by the t-transitions from w_1 to w_2 and back, whereas the reflexive arrows are, in a sense, less expected in the given scenario. Therefore, it becomes important to single out those executions of the action that are deemed more normal from those that are not. In Figure 1, this would amount to enriching the semantic structure (in a way still to be defined) with information specifying that the pairs (w_1, w_2) and (w_2, w_1) somehow take 'precedence' over (w_1, w_1) and (w_2, w_2) when reasoning about possible executions of the action.

Let us now consider a variant of the above scenario (although just as simple), in which we have only one atomic proposition, correct (which we shall abbreviate as c), the intuition of which is that a proposed proof for a mathematical statement is correct. Furthermore, let us assume there is a good mathematician, M, whose knowledge about the correctness of the proof is of interest to us. Figure 2 below depicts one possible configuration of this scenario.

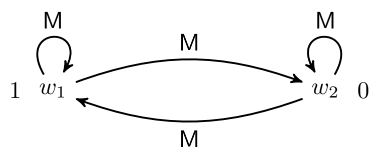

Figure 2: A possible-worlds model for one agent (the good mathematician M) and one atom (correct).

As a good mathematician, Agent M should know whether the proof is correct. Nevertheless, in the model of Figure 2, M does not know (in the classical sense) whether the proof is correct or not, since, also by virtue of being a good mathematician, M admits the (unlikely) possibility of being wrong (at least until the proof has been submitted to peer-reviewed scrutiny). In this case, we would say that Agent M *defeasibly knows* whether the proof is correct or not, an epistemic stance that can be adopted by 'focusing' on the most normal (or expected) of the epistemic possibilities held by the agent, namely (w_1, w_1) and (w_2, w_2) in Figure 2, which, in this example, are more normal than (w_1, w_2) and (w_2, w_1). (It is not hard to see that the motivation above also holds in a doxastic context, as certain beliefs may be more entrenched than others.)

In order to motivate the foregoing ideas in a deontic context, let us assume a language with a single propositional atom, namely fair-play, henceforth abbreviated f and of which the intuition is that, in a competition, the players abide by an established standard of 'decency' or an 'honorable conduct'. In this context, adopting a fair-play stance is not to be seen as an obligation in the usual (strict) meaning of the term. It is rather a matter of best practice in that it corresponds to the expected, though not enforceable (even if, in some cases, liability-biding), attitude. Figure 3 below depicts a possible-worlds model for this scenario.

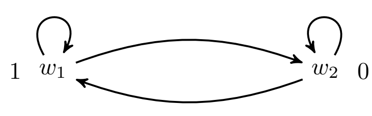

Figure 3: A possible-worlds model for one atom (fair-play) in a deontic-context.

A case can be made that envisioning an f-world as a better alternative (to the current one) is more appropriate than the contemplation of a $\neg$f-one. Semantically, this requirement would be translated as setting the pairs (w_1, w_1) and (w_2, w_1) as more preferred than (w_2, w_2) and (w_1, w_2). In this specific example, it happens that we could also model the underlying preference as an ordering on worlds, with f-worlds preferred to the $\neg$f-worlds. However, in the preceding examples, ordering worlds rather than pairs of worlds is neither intuitive nor is it immediately clear whether this is even possible.

In this work, we address precisely these issues. We shall start by shifting the normality spotlight from possible worlds to transitions amongst them, i.e., to accessibility relations in Kripke frames. The justification for doing so stems from a comparison with the classical (monotonic) case: In classical Kripkean semantics, modalities are primarily about accessibility, only secondarily about worlds' contents. Hence, we contend that accounts of a notion of defeasibility in modalities (like those illustrated above) should primarily focus on normality of the accessibility relations rather than (or at least prior to) that of the (accessible) worlds. With that we hope to pave the way for further explorations of non-monotonicity in modal logics, in particular in extensions of the preferential approach therein (Britz, Meyer, and Varzinczak 2011a).

Preliminaries

In this section, we provide the required formal background for the rest of this work. In particular, we set up the notation and conventions that shall be followed in the upcoming sections. (The reader conversant with modal logic can safely skip the first subsection below.)

Modal Logic

We work in a set of *atomic propositions* $\mathcal{P}$, using the logical connectives $\wedge$ (conjunction), $\neg$ (negation), and a set of modal operators $\Box_i$, $1 \leqslant i \leqslant n$. Propositions are denoted by $p, q, \ldots$, and sentences by $\alpha, \beta, \ldots$, constructed in the usual way according to the rule ($1 \leqslant i \leqslant n$):

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box_i\alpha$$

All the other truth-functional connectives ($\vee$, $\rightarrow$, $\leftrightarrow$, ...) are defined in terms of $\neg$ and $\wedge$ in the usual way. Given $\Box_i$, $1 \leqslant i \leqslant n$, with $\Diamond_i$ we denote its *dual* modal operator, i.e., for any α, $\Diamond_i\alpha := \neg\Box_i\neg\alpha$. We use $\top$ as an abbreviation for $p \vee \neg p$ and $\bot$ as an abbreviation for $p \wedge \neg p$, for some $p \in \mathcal{P}$. With $\mathcal{L}^{\Box}$ we denote the set of all sentences of the modal language.

The semantics is the standard possible-worlds one:

Definition 1 (Kripke Model) *A Kripke model is a tuple $\mathscr{M} := \langle W, R, V \rangle$ where W is a (non-empty) set of possible worlds, $R := \langle R_1, \ldots, R_n \rangle$, where each $R_i \subseteq W \times W$ is an accessibility relation on W, $1 \leqslant i \leqslant n$, and $V : W \longrightarrow \{0,1\}^{\mathcal{P}}$ is a valuation function mapping possible worlds into propositional valuations.*

As an example, Figure 4 depicts the Kripke model $\mathscr{M}_1 = \langle W_1, R_1, V_1 \rangle$, where $W_1 := \{w_i \mid 1 \leqslant i \leqslant 4\}$, $R_1 := \langle R_a, R_b \rangle$, with $R_a := \{(w_1, w_2), (w_1, w_3), (w_4, w_3)\}$, and $R_b := \{(w_1, w_4), (w_2, w_3)\}$, and V_1 is the obvious valuation function.

In our pictorial representations of models, we represent propositional valuations as sequences of 0s and 1s, and with the obvious implicit ordering of atoms. Thus, for the logic generated from p and q, the valuation in which p is true and q is false will be represented as 10.

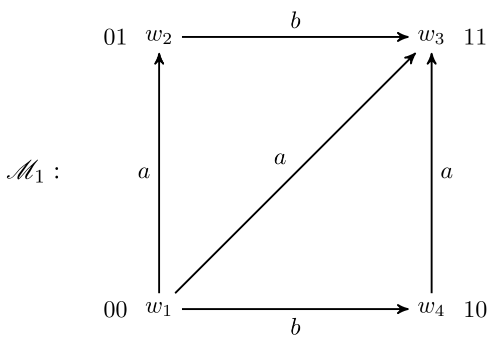

Figure 4: A Kripke model for $\mathcal{P} = \{p, q\}$ and two modalities, namely a and b.

We shall use $w, u, v, \ldots$ (possibly decorated with primes) to denote possible worlds. Moreover, where it aids readability, we shall henceforth sometimes write tuples of the form (w, w') as ww'.

Sentences of $\mathcal{L}^{\Box}$ are true or false relative to a possible world in a given Kripke model:

Definition 2 (Truth Conditions) *Let $\mathscr{M} = \langle W, R, V \rangle$ and $w \in W$:*

- $\mathscr{M}, w \Vdash p$ *if and only if* $V(w)(p) = 1$;
- $\mathscr{M}, w \Vdash \neg\alpha$ *if and only if* $\mathscr{M}, w \nVdash \alpha$;
- $\mathscr{M}, w \Vdash \alpha \wedge \beta$ *if and only if* $\mathscr{M}, w \Vdash \alpha$ *and* $\mathscr{M}, w \Vdash \beta$;
- $\mathscr{M}, w \Vdash \Box_i\alpha$ *if and only if* $\mathscr{M}, w' \Vdash \alpha$ *for all* w' *such that* $(w, w') \in R_i$.

Given $\alpha \in \mathcal{L}^{\Box}$ and $\mathscr{M} = \langle W, R, V \rangle$, we say that $\mathscr{M}$ *satisfies* α if there is at least one world $w \in W$ such that $\mathscr{M}, w \Vdash \alpha$. We say that $\mathscr{M}$ is a *model of* α (alias α is *true* in $\mathscr{M}$), denoted $\mathscr{M} \Vdash \alpha$, if $\mathscr{M}, w \Vdash \alpha$ for every world $w \in W$. Given a class (i.e., a collection) of models $\mathcal{M}$, we say that α is *valid* in $\mathcal{M}$, denoted $\models_{\mathcal{M}} \alpha$, if and only if every Kripke model $\mathscr{M} \in \mathcal{M}$ is a model of α. Given $\mathcal{K} \subseteq \mathcal{L}^{\Box}$ and $\alpha \in \mathcal{L}^{\Box}$, we say that $\mathcal{K}$ *locally entails* α in the class of models $\mathcal{M}$, denoted $\mathcal{K} \models_{\mathcal{M}} \alpha$, if and only if for every Kripke model $\mathscr{M} \in \mathcal{M}$ and every w in $\mathscr{M}$, if $\mathscr{M}, w \Vdash \beta$ for every $\beta \in \mathcal{K}$, then $\mathscr{M}, w \Vdash \alpha$. (When the class of models we are working with is clear from the context, we shall dispense with subscripts and just write $\models \alpha$ and $\mathcal{K} \models \alpha$.)

Here we shall assume the system of normal modal logic K, of which all the other normal modal logics are extensions. Semantically, K is characterised by the class of all Kripke models (Definition 1). Syntactically, K corresponds to the smallest set of sentences containing all propositional tautologies, all instances of the axiom schema $\mathsf{K} : \Box_i(\alpha \rightarrow \beta) \rightarrow (\Box_i\alpha \rightarrow \Box_i\beta)$, $1 \leqslant i \leqslant n$, and closed under the *rule of necessitation* below:

$$(\text{RN})\ \frac{\alpha}{\Box_i\alpha} \tag{1}$$

For more details on modal logic, we refer the reader to the handbook by Blackburn *et al*. (2006).

Preferential Modalities

In previous work (Britz, Meyer, and Varzinczak 2011a; Britz and Varzinczak 2013), we have investigated the fruitfulness of extending the standard Kripke semantics with a preference relation on the set of possible worlds. This gives rise to the following semantic structure, of which the underlying motivation is similar to that behind Boutilier's (1994) CT4O models and the plausibility models of Baltag and Smets (2006; 2008).

Definition 3 (*W*-Ordered Model) *A W-ordered model is a tuple $\mathscr{W} := \langle W, R, V, \prec \rangle$ where $\langle W, R, V \rangle$ is as in Definition 1 and $\prec \subseteq W \times W$ is a well-founded strict partial order on W, i.e., $\prec$ is irreflexive, transitive and every non-empty $W' \subseteq W$ has minimal elements w.r.t. $\prec$ (see Definition 4).*

The intuition behind the preference relation $\prec$ in a *W*-ordered model $\mathscr{W}$ is that the worlds lower down in the ordering are deemed as more preferred (or more normal) than those higher up.

Definition 4 (Minimality w.r.t. $\prec$) *Let $\mathscr{W} = \langle W, R, V, \prec \rangle$ be a W-ordered model and let $X \subseteq W$. Then $\min_{\prec} X := \{w \in X \mid$ there is no $w' \in X$ such that $w' \prec w\}$, i.e., $\min_{\prec} X$ denotes the minimal elements of X with respect to the preference relation $\prec$.*

As an example, Figure 5 below depicts the *W*-ordered model $\mathscr{W}_1 = \langle W_1, R_1, V_1, \prec_1 \rangle$, where $\langle W_1, R_1, V_1 \rangle$ is as in Figure 4 and $\prec_1 := \{(w_1, w_2), (w_2, w_3), (w_1, w_3), (w_4, w_3)\}$.

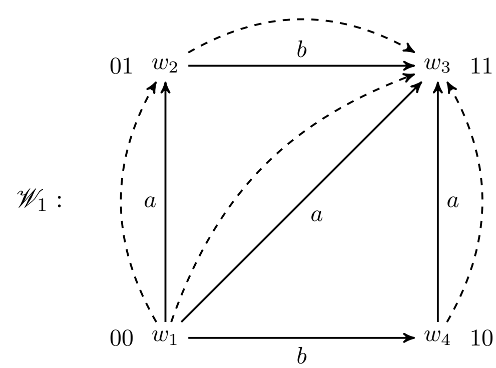

Figure 5: A *W*-ordered model for $\mathcal{P} = \{p, q\}$ and two modalities (a and b). The preference relation $\prec_1$ is represented by the dashed arrows, which point from more preferred to less preferred worlds.

We can then extend $\mathcal{L}^{\Box}$ with a family of defeasible modal operators $\mathrel{\Box\!\!\!\sim}_i$ (called 'flag'), $1 \leqslant i \leqslant n$, where n is the number of classical modalities in the language. The sentences of the extended language are then recursively defined by:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box_i\alpha \mid ⚐_i\alpha$$

As before, the other connectives are defined in terms of $\neg$ and $\wedge$ in the usual way, $\top$ and $\bot$ are seen as abbreviations, and $\Diamond_i$ is the dual of $\Box_i$. Moreover, with $⟡_i$ (called 'flame') we denote the dual of $⚐_i$. We shall use $\mathcal{L}^{⚐}$ to denote the set of all sentences of such a richer language.

Definition 5 (Truth Conditions for $\mathcal{L}^{⚐}$) *Let a W-ordered model $\mathscr{W} = \langle W, R, V, \prec\rangle$ and let $w \in W$.*

- *$\mathcal{L}^{\Box}$-sentences are evaluated as usual (Definition 2);*
- *$\mathscr{W}, w \Vdash ⚐_i\alpha$ if and only if for all w', if $w' \in \min_{\prec} R_i(w)$, then $\mathscr{W}, w' \Vdash \alpha$.*

The notions of satisfaction, truth (in a model), validity (in a class of models) and local entailment are generalised to $\mathcal{L}^{⚐}$-sentences and *W*-ordered models in the obvious way.

Informally, a sentence of the form $⚐_i\alpha$ holds in a world if α holds in all the most preferred amongst its i-accessible worlds. It is easy to see that $⚐$ is weaker than $\Box$, i.e., the following is a validity ($1 \leqslant i \leqslant n$):

$$\models \Box_i\alpha \rightarrow ⚐_i\alpha$$

Hence, intuitively, flag can be read as *defeasible necessity*.

As an example, considering the *W*-ordered model $\mathscr{W}_1$ from Figure 5, we have that $\mathscr{W}_1, w_1 \Vdash ⚐_a\neg p$ (but note that $\mathscr{W}_1, w_1 \nVdash \Box_a\neg p$).

Revisiting Preferential Modal Logics

In spite of its gain in expressiveness when checked against traditional approaches to defeasible reasoning, $⚐$ does not quite seem to allow us to formalise the type of reasoning motivated in the Introduction inasmuch as it relies on orderings on worlds. In this section, we shall revisit the framework for preferential modalities, in particular its semantic constructions.

R-Ordered Models

We start by giving a formal account of the semantic ideas put forward in the Introduction.

Definition 6 (*R*-Ordered Model) *An R-ordered model is a tuple $\mathscr{R} := \langle W, R, V, \ll\rangle$ where W is a (non-empty and possibly infinite) set of possible worlds, $R := \langle R_1, \ldots, R_n\rangle$, where each $R_i \subseteq W \times W$ is an accessibility relation on W, for $1 \leqslant i \leqslant n$, $V : W \longrightarrow \{0,1\}^{\mathcal{P}}$ is a valuation function assigning each world to a valuation on $\mathcal{P}$, and $\ll := \langle \ll_1, \ldots, \ll_n\rangle$, where each $\ll_i \subseteq R_i \times R_i$, for $1 \leqslant i \leqslant n$, is a well-founded strict partial order on the respective R_i, i.e., each $\ll_i$ is irreflexive, transitive and every non-empty $R'_i \subseteq R_i$ has minimal elements w.r.t. $\ll_i$ (see Definition 7).*

Given $\mathscr{R} := \langle W, R, V, \ll\rangle$, the intuition of *W*, *R* and *V* is the same as that in a standard Kripke model. The intuition of each $\ll_i$ in $\ll$ is that the pairs (w, w') that are lower down in the ordering $\ll_i$ are deemed as the most normal (or typical, or expected) in the context of R_i.

Definition 7 (Minimality w.r.t. $\ll_i$) *Let $\mathscr{R} = \langle W, R, V, \ll\rangle$ be an R-ordered model and let $X \subseteq R_i$, for some $1 \leqslant i \leqslant n$. Then $\min_{\ll_i} X := \{(w, w') \in X \mid$ there is no $(u, v) \in X$ such that $(u, v) \ll_i (w, w')\}$, i.e., $\min_{\ll_i} X$ denotes the minimal elements of X with respect to the preference relation $\ll_i$ associated to R_i.*

Since we assume each $\ll_i$ to be a well-founded strict partial order on the respective R_i, we are guaranteed that for every $X \subseteq R_i$ such that $X \neq \emptyset$, $\min_{\ll_i} X$ is well defined.

As an example, Figure 6 below depicts the *R*-ordered model $\mathscr{R}_1 := \langle W_1, R_1, V_1, \ll_1\rangle$, where $\langle W_1, R_1, V_1\rangle$ is as in Figure 4, and $\ll_1 := \langle \ll_a, \ll_b\rangle$, where $\ll_a := \{(w_1w_2, w_1w_3), (w_1w_3, w_4w_3), (w_1w_2, w_4w_3)\}$ and $\ll_b := \{(w_1w_4, w_2w_3)\}$, represented, respectively, by the dashed and the dotted arrows in the picture. (Note the direction of the $\ll$-arrows, which point from more preferred to less preferred transitions.) For the sake of readability, in our pictorial representations of *R*-ordered models, we shall omit the transitive $\ll$-arrows.

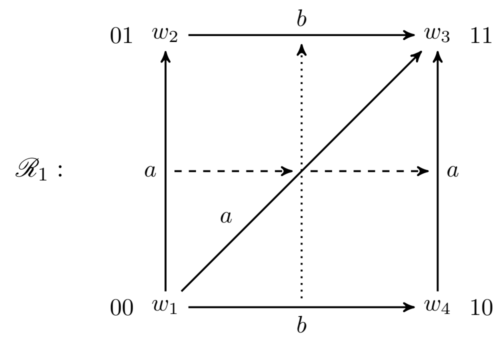

Figure 6: An *R*-ordered model for $\mathcal{P} = \{p, q\}$ and two modalities. The preference relation $\ll_a$ is represented by the dashed arrows, whereas $\ll_b$ by the dotted one.

A New Logic of Defeasible Modalities

We shall now enrich our underlying modal language with a family of additional modal operators $\approxident_i$, $1 \leqslant i \leqslant n$, where n is the number of classical modalities in the language. (For lack of a better term, we shall call $\approxident$ the 'banner'.) The sentences of the extended modal language are recursively defined as follows:

$$\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box_i\alpha \mid \approxident_i\alpha$$

With $\mathcal{L}^{\approxident}$ we shall denote the set of all sentences of the banner language.

Definition 8 *Let $\mathscr{R} = \langle W, R, V, \ll\rangle$. For every $w \in W$ and every $R_i \subseteq W \times W$, we define:*

$$R_i^w := \{(u, v) \mid (u, v) \in R_i \text{ and } u = w\}$$

Definition 9 ($\mathcal{L}^{\approxident}$ Truth Conditions) *Let $\mathscr{R} = \langle W, R, V, \ll\rangle$ be an R-ordered model and $w \in W$.*

- *$\mathcal{L}^{\Box}$-sentences are evaluated as usual;*
- *$\mathscr{R}, w \Vdash \approxident_i\alpha$ if and only if for every w', if $(w, w') \in \min_{\ll_i} R_i^w$, then $\mathscr{R}, w' \Vdash \alpha$.*

The notions of satisfaction, truth (in a model), validity (in a class of models) and local entailment are also generalised to $\mathcal{L}^{\approxeq}$-sentences and *R*-ordered models in the usual way.

Informally, a sentence of the form $\approxeq_i\alpha$ holds in a world if α holds in all its most normally i-accessible worlds. As an example, in the *R*-ordered model $\mathscr{R}_1$ of Figure 6, we have that $\mathscr{R}_1, w_1 \Vdash \approxeq_a\neg p$ (but, of course, $\mathscr{R}_1, w_1 \not\Vdash \square_a\neg p$).

Incidentally, $\approxeq$ too is weaker than $\square$, as witnessed by the validity below ($1 \leqslant i \leqslant n$):

$$\models \square_i\alpha \rightarrow \approxeq_i\alpha$$

Hence, $\approxeq$ provides an alternative perspective on the notion of defeasible necessity as formalised by $\overset{\approx}{\sqcup}$. For instance, in an action context, some executions (which refer to transitions) of a given action are deemed as more normal than others. A priori, this is different from saying that some effects (which refer to target worlds) are normal. Indeed, an abnormal execution may still lead to the expected (normal) effect, just as a normal execution may produce an abnormal effect. (We shall come back to this issue later on.)

The definitions of *R*-ordered models and $\approxeq$, alongside the comment right above, raise the question as to how $\mathcal{L}^{\overset{\approx}{\sqcup}}$ and $\mathcal{L}^{\approxeq}$ compare to each other in terms of expressive power. This is what we address in the next section.

Preferential Bisimulations

Standard bisimulations are used to determine whether two Kripke models have the same modal properties, and to reason about modal expressivity. Here, we extend the definition of bisimulations to *W*-ordered and *R*-ordered models, and use it to make precise the connection between these notions, and the resulting modalities and modal languages.

Definition 10 *Let $\mathscr{M} = \langle W, R, V\rangle$ and $\mathscr{M}' = \langle W', R', V'\rangle$. A bisimulation between $\mathscr{M}$ and $\mathscr{M}'$ is a non-empty binary relation E between their domains (that is, $\mathsf{E} \subseteq W \times W'$) such that, whenever $w\mathsf{E}w'$, we have that:*

1. *For every $p \in \mathcal{P}$, $\mathscr{M}, w \Vdash p$ if and only if $\mathscr{M}', w' \Vdash p$;*
2. *if wR_iv, then there exists a world v' in W' such that $v\mathsf{E}v'$ and $w'R'_iv'$, and*
3. *if $w'R'_iv'$, then there exists a world v in W such that $v\mathsf{E}v'$ and wR_iv.*

Informally, two worlds are bisimilar if they satisfy the same atomic information, and their modal accessibility structures match. Two pointed models $(\mathscr{M}, w)$ and $(\mathscr{M}', w')$ are bisimilar if there exists a bisimulation E between $\mathscr{M}$ and $\mathscr{M}'$ such that $w\mathsf{E}w'$. It then follows that:

Lemma 1 (Bisimulation invariance lemma) *If E is a bisimulation between $\mathscr{M} = \langle W, R, V\rangle$ and $\mathscr{M}' = \langle W', R', V'\rangle$, $w \in W$ and $w' \in W'$, and $w\mathsf{E}w'$, then w and w' satisfy the same basic modal sentences.*

The next definition and lemma generalise bisimulations to take account of a preference order on worlds, as defined on models of $\mathcal{L}^{\overset{\approx}{\sqcup}}$. Informally, two worlds are bisimilar if they satisfy the same atomic information and their modal accessibility structures match, both with respect to accessible worlds and with respect to most preferred relative accessible worlds. Bisimilar worlds then also satisfy the same preferential modal sentences.

Definition 11 (*W*-ordered bisimulation) *Let W-ordered models $\mathscr{W} = \langle W, R, V, \prec\rangle$ and $\mathscr{W}' = \langle W', R', V', \prec'\rangle$. A bisimulation between $\mathscr{W}$ and $\mathscr{W}'$ is a non-empty binary relation $\mathsf{E} \subseteq W \times W'$ such that, whenever $w\mathsf{E}w'$, we have that:*

1. *For every $p \in \mathcal{P}$, $\mathscr{W}, w \Vdash p$ if and only if $\mathscr{W}', w' \Vdash p$;*
2. *if wR_iv, then there exists a world v' in W' such that $v\mathsf{E}v'$ and $w'R'_iv'$, and*
   - *if $v \in \min_{\prec} R_i(w)$, then $v' \in \min_{\prec'} R'_i(w')$;*
3. *if $w'R'_iv'$, then there exists a world v in W such that $v\mathsf{E}v'$ and wR_iv, and*
   - *if $v' \in \min_{\prec'} R'_i(w')$, then $v \in \min_{\prec} R_i(w)$.*

Lemma 2 (*W*-ordered bisimulation invariance lemma) *If E is a bisimulation between $\mathscr{W} = \langle W, R, V, \prec\rangle$ and $\mathscr{W}' = \langle W', R', V', \prec'\rangle$, and $w\mathsf{E}w'$, then w and w' satisfy the same modal sentences in the extended modal language $\mathcal{L}^{\overset{\approx}{\sqcup}}$.*

Proof:
The lemma is proved by structural induction on $\alpha \in \mathcal{L}^{\overset{\approx}{\sqcup}}$. We show that, for any $w \in W$ and $w' \in W'$, if $w\mathsf{E}w'$, then $\mathscr{W}, w \Vdash \alpha$ iff $\mathscr{W}', w' \Vdash \alpha$. For atomic propositions, and when $\alpha = \neg\beta$ or $\alpha = \beta_1 \vee \beta_2$, the proof is immediate. We consider the remaining two cases, namely when $\alpha = \square_i\beta$ or $\alpha = \overset{\approx}{\sqcup}_i\beta$.

Assume $\alpha = \square_i\beta$ and let $\mathscr{W}, w \Vdash \square_i\beta$. The proof is as for basic modal logic: Suppose $v' \in R'_i(w')$. Since $w\mathsf{E}w'$, there is some $v \in R_i(w)$ with $v\mathsf{E}v'$. Therefore $\mathscr{W}, v \Vdash \beta$, and hence $\mathscr{W}', v' \Vdash \beta$ by the induction hypothesis. It follows that $\mathscr{W}', w' \Vdash \square_i\beta$. A symmetric argument applies if $\mathscr{W}', w' \Vdash \square_i\beta$.

Assume $\alpha = \overset{\approx}{\sqcup}_i\beta$ and let $\mathscr{W}, w \Vdash \overset{\approx}{\sqcup}_i\beta$. Suppose $v' \in \min_{\prec} R'_i(w')$. Since $w\mathsf{E}w'$, there is some $v \in \min_{\prec} R_i(w)$ with $v\mathsf{E}v'$. Therefore $\mathscr{W}, v \Vdash \beta$, and hence $\mathscr{W}', v' \Vdash \beta$ by the induction hypothesis. It follows that $\mathscr{W}', w' \Vdash \overset{\approx}{\sqcup}_i\beta$. A symmetric argument applies if $\mathscr{W}', w' \Vdash \square_i\beta$. ■

We now turn to bisimulations between *R*-ordered models. As above, two worlds are bisimilar if they satisfy the same atomic information and their modal accessibility structures match, both in terms of accessible worlds and in terms of preference of accessibility.

Definition 12 (*R*-ordered bisimulation) *Let R-ordered models $\mathscr{R} = \langle W, R, V, \lll\rangle$ and $\mathscr{R}' = \langle W', R', V', \lll'\rangle$. A bisimulation between $\mathscr{R}$ and $\mathscr{R}'$ is a non-empty binary relation $\mathsf{E} \subseteq W \times W'$ such that, whenever $w\mathsf{E}w'$, we have that:*

1. *For every $p \in \mathcal{P}$, $\mathscr{R}, w \Vdash p$ if and only if $\mathscr{R}', w' \Vdash p$;*
2. *if wR_iv, then there exists a world v' in W' such that $v\mathsf{E}v'$ and $w'R'_iv'$, and*
   - *if $wv \in \min_{\lll_i} R_i^w$, then $w'v' \in \min_{\lll'_i} R_i'^{w'}$;*
3. *if $w'R'_iv'$, then there exists a world v in W such that $v\mathsf{E}v'$ and wR_iv, and*
   - *if $w'v' \in \min_{\lll'_i} R_i'^{w'}$, then $wv \in \min_{\lll_i} R_i^w$.*

Lemma 3 (*R*-ordered bisimulation invariance lemma)
If E *is a bisimulation between* $\mathscr{R} = \langle W, R, V, \ll \rangle$ *and* $\mathscr{R}' = \langle W', R', V', \ll' \rangle$, $w \in W$ *and* $w' \in W'$, *and* $w\mathsf{E}w'$, *then* w *and* w' *satisfy the same modal sentences in the extended language* $\mathcal{L}^{\thickapprox}$.

Proof:
The proof is by structural induction on $\alpha \in \mathcal{L}^{\thickapprox}$ and is similar to that of Lemma 2. We show that, for any $w, w' \in W$, if $w\mathsf{E}w'$, then $\mathscr{R}, w \Vdash \alpha$ iff $\mathscr{R}', w' \Vdash \alpha$. We only prove the case when $\alpha = \thickapprox_i \beta$.

Assume $\alpha = \thickapprox_i \beta$ and let $\mathscr{R}, w \Vdash \thickapprox_i \beta$. Suppose $w'v' \in \min_{\ll'_i} R_i'^{w'}$. Since $w\mathsf{E}w'$, there is some $wv \in \min_{\ll_i} R_i^w$ with $v\mathsf{E}v'$. Therefore $\mathscr{R}, v \Vdash \beta$, and hence $\mathscr{R}', v' \Vdash \beta$ by the induction hypothesis. It follows that $\mathscr{R}', w' \Vdash \thickapprox_i \beta$. A symmetric argument applies if $\mathscr{R}', w' \Vdash \thickapprox_i \beta$. ■

The relationship between $\mathcal{L}^{\thickapprox}$ and $\mathcal{L}^{\approxeq}$, and between *R*-ordered and *W*-ordered models, can be made precise using bisimulations. We first show that $\mathcal{L}^{\thickapprox}$ is at least as expressive as $\mathcal{L}^{\approxeq}$: Given a sentence $\alpha \in \mathcal{L}^{\approxeq}$, let $\alpha^{\thickapprox}$ be the sentence obtained by replacing all occurrences of $\approxeq_i$ in α with $\thickapprox_i$.

Definition 13 *Let* $\mathscr{W} = \langle W, R, V, \prec \rangle$ *be a W-ordered model. For any* $u, v, w \in W$ *such that* wR_iu *and* wR_iv *and* $u \prec v$, *let* $wu \ll_i wv$. *Then* $\mathscr{R}_{\mathscr{W}} = \langle W, R, V, \ll \rangle$ *is the R-ordered model induced by* $\mathscr{W}$.

Lemma 4 *For any* $\alpha \in \mathcal{L}^{\approxeq}$, $\mathscr{W} = \langle W, R, V, \prec \rangle$ *and* $w \in W$, $\mathscr{W}, w \Vdash \alpha$ *if and only if in the R-ordered model* $\mathscr{R}_{\mathscr{W}} = \langle W, R, V, \ll \rangle$ *induced by* $\mathscr{W}$, $\mathscr{R}_{\mathscr{W}}, w \Vdash \alpha^{\thickapprox}$.

Proof:
The proof is simple and proceeds by structural induction on the sentence α. ■

Lemma 4 shows that, if α and β are not equivalent in $\mathcal{L}^{\approxeq}$, then their translations $\alpha^{\thickapprox}$ and $\beta^{\thickapprox}$ are also not equivalent in $\mathcal{L}^{\thickapprox}$. Further, if $(\mathscr{W}, w)$ and $(\mathscr{W}', w')$ are distinguishable by some $\alpha \in \mathcal{L}^{\approxeq}$, say, $\mathscr{W}, w \Vdash \alpha$ and $\mathscr{W}', w' \nVdash \alpha$, then $\mathscr{R}_{\mathscr{W}}$ and $\mathscr{R}'_{\mathscr{W}}$ are distinguishable by $\alpha^{\thickapprox} \in \mathcal{L}^{\thickapprox}$. Hence, $\mathcal{L}^{\thickapprox}$ is at least as expressive as $\mathcal{L}^{\approxeq}$.

The converse of this result may not be as obvious to see, and translating *R*-ordered models to *W*-ordered models requires more care. The light switch example (Figure 1) shows that, even in the case of a single modality, there is no direct translation of a preference order on *R* to a preference order on *W*. There is no order on the two worlds w_1 and w_2 such that w_1 is the preferred result of toggling the light switch when the light is off, but w_2 is the preferred result when the light is on. A further problematic aspect is that *R*-ordered models allow for a preference order on each accessibility relation, whereas a *W*-ordered semantics assume a single common preference order on worlds.

Definition 14 *Let* $\mathscr{R} = \langle W, R, V, \ll \rangle$ *be an R-ordered model with single accessibility relation* R_1. *Let* $W' = W \times W$; *let* $V'(uw) = V(w)$; *let* $uvR_1'vw$ *whenever* vR_1w, *and let* $uv \prec u'v'$ *whenever* $uv \ll u'v'$. *Then* $\mathscr{W}_{\mathscr{R}} = \langle W', R', V', \prec \rangle$ *is the W-ordered model induced by* $\mathscr{R}$.

As an example, we apply Definition 14 to obtain the *W*-ordered models induced by the models of Figures 1 and 2, and depicted in Figure 7 and Figure 8 respectively. Note that in Figure 7, $w_1w_2 \prec w_1w_1$ and $w_2w_1 \prec w_2w_2$, reflecting the intuition of normal execution of the action as an order on worlds. In Figure 8, the order on worlds is reversed, with $w_1w_1 \prec w_1w_2$ and $w_2w_2 \prec w_2w_1$, depicting the intuition of defeasible knowledge of the agent as an order on worlds.

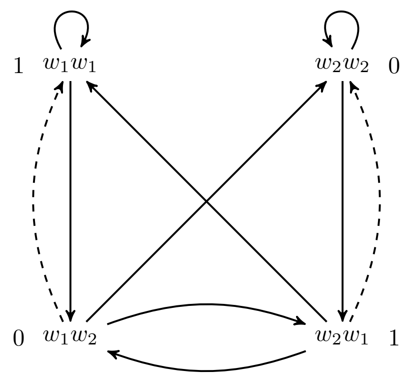

Figure 7: The induced *W*-ordered possible-worlds model for one action (toggle) and one atom (on).

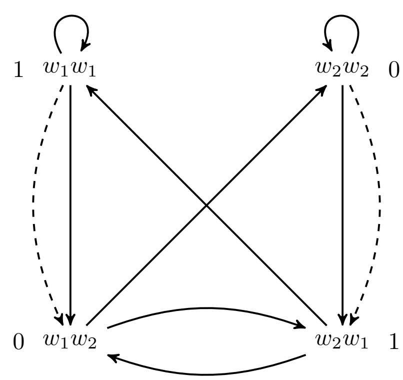

Figure 8: The induced *W*-ordered possible-worlds model for one agent (M) and one atom (correct).

Theorem 1 *Let* $\mathscr{R} = \langle W, R, V, \ll \rangle$ *be an R-ordered model with a single accessibility relation* R_1 *and let* $\mathscr{W}_{\mathscr{R}} = \langle W, R, V, \prec \rangle$ *be the W-ordered model induced by* $\mathscr{R}$. *Let* $\mathscr{R}_{\mathscr{W}_{\mathscr{R}}} = \langle W', R', V', \ll' \rangle$ *be the R-ordered model induced by* $\mathscr{W}_{\mathscr{R}}$. *Then there is a full bisimulation between* $\mathscr{R}$ *and* $\mathscr{R}_{\mathscr{W}_{\mathscr{R}}}$, *i.e., with domain W and range* $W \times W$.

Proof:
Let E be defined by: $w\mathsf{E}vw$ for all $v, w \in W$. We need to show that E is a full bisimulation relation. So, let $u, v \in W$. Then $v\mathsf{E}uv$.

1. It follows immediately from the construction of $\mathscr{R}_{\mathscr{W}_{\mathscr{R}}}$ that v and uv satisfy the same atomic propositions.
2. Suppose vR_1w. It follows again from the construction of $\mathscr{R}_{\mathscr{W}_{\mathscr{R}}}$ that $uvR_1'vw$ and $w\mathsf{E}vw$. Further, if $vw \in \min_{\ll_1} R_1^v$, then $vw \in \min_{\prec} R_1(uv)$, and hence $vw \in \min_{\ll'_1} (R_1')^{uv}$.

3. Suppose uvR'_1vw. It again follows from the construction of $\mathscr{R}_{\mathscr{W}_{\mathscr{R}}}$ that vR_1w and $w\mathsf{E}vw$. Further, if $vw \in \min_{\ll'_1}(R'_1)^{uv}$, then $vw \in \min_{<} R_1(w)$, and hence $vw \in \min_{\ll_1} R_1^v$.

■

We illustrate the construction of Theorem 1 by applying Definition 13 to the induced *W*-ordered model in Figure 7 to obtain the *R*-ordered model of Figure 9. In Figure 9, the dashed arrows represent the preference order $\ll'$. Theorem 1 then states that the *R*-ordered model of Figure 1 (with the order as described in the Introduction) is bisimilar to the *R*-ordered model of Figure 9, The construction is via the *W*-ordered model of Figure 7.

Similarly, the *R*-ordered model of Figure 2 (again, with the order as described in the Introduction) is bisimilar to the *R*-ordered model of Figure 10, which is constructed via the *W*-ordered model of Figure 8.

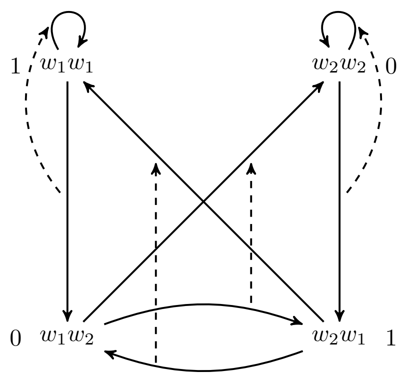

Figure 9: The induced bisimilar *R*-ordered model for one action (toggle) and one atom (on).

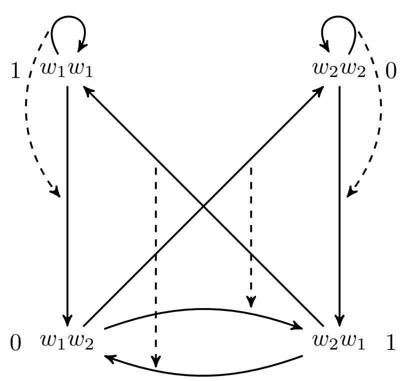

Figure 10: The induced bisimilar *R*-ordered model for one agent (M) and one atom (correct).

Corollary 1 *$\mathcal{L}^{\approxeq}$ and $\mathcal{L}^{\approx}$ can distinguish between the same modal propositions when restricted to a single modality.*

Proof:
The bisimulation result of Theorem 1 shows that any *R*-ordered model is bisimilar to some *R*-ordered model induced by a *W*-ordered model. Lemma 3 ensures that bisimilar worlds satisfy the same modal sentences, and that bisimilar models can distinguish between the same modal properties. We need therefore consider only *R*-ordered models induced by some *W*-ordered model when reasoning about expressivity. The result then follows from Lemma 4. ■

Corollary 1 may be seen as a negative result in the sense that, at least in the monomodal case, no richer language is obtained when substituting a preference order on the accessibility relation for the preference order on worlds. It is also clear that the results of Theorem 1 and Corollary 1 can be generalised to multi-modal languages if multiple preference relations on *W* are allowed.

What, then, has been gained? As we have argued, there are a number of contexts in which an order on the accessibility relation has an intuitive appeal. The induced *W*-ordered models of Definition 13 are technically useful, but intuitively hard to motivate. However, from an implementation perspective, we now know that a reasoner based on a *W*-ordered semantics suffices also for reasoning over *R*-ordered models. This, together with our previous results (Britz and Varzinczak 2013), establish the following:

Corollary 2 *Satisfiability checking for monomodal $\mathcal{L}^{\approxeq}$ is* PSPACE*-complete.*

Discussion and Related Work

It might be worth emphasising that the logics we have investigated here do not aim at providing a formal account of the notion of *most*, as addressed in the study of generalised quantifiers (Lindström 1966) and, more recently, in a modal context by Veloso *et al*. (2009) and Askounis *et al*. (2012). Clearly, they are not about degrees of truth, as it has been studied in fuzzy logics, nor about degrees of possibility and necessity, as addressed by possibilistic logics (Dubois, Lang, and Prade 1994). Instead, here we have investigated a rather complementary notion to those ones, namely that of *normal*, *expected*, *practical* necessity, which need not rely on majority or degrees of likelihood.

In a sense, the notions we investigated here can be seen as the qualitative counterpart of possibilistic modalities (Liau 1999; Liau and Lin 1996). (We thank an anonymous referee for pointing this out to us.) There, each possible world w is associated with a *possibility distribution* $\pi_w : W \longrightarrow [0, 1]$, the intuition of which is to capture the degree of likelihood (in terms of belief) of all possible worlds w.r.t. w. In that setting, the pairs (w, w') for which $\pi_w(w')$ is maximal correspond here to the most preferred pairs in a single accessibility relation. In this sense, there are strong links between *monomodal* $\eqsim$ and the preferential possibilistic semantics for epistemic reasoning.

Currently, the definition of *R*-ordered model (Definition 6) allows only for elements of the same accessibility relation R_i to be ordered (via the respective $\ll_i$). More generally, we could have defined $\ll$ as a relation on $\bigcup_{1\leqslant i\leqslant n} R_i \times$

$\bigcup_{1 \leqslant i \leqslant n} R_i$, so that we allow pairs (w, w') belonging to different *R*-components to be compared as well. An investigation of the philosophical and practical ramifications of this alternative definition is left for future work.

We have seen that one can obtain *R*-ordered models from *W*-ordered models by inducing an ordering on edges from the ordering on worlds. The result is an 'embedding' of $\backsimeq$ into $\mathcal{L}^{\approx}$. Conversely, in the monomodal case, we can obtain *W*-ordered models from *R*-ordered models by inducing an ordering on worlds from an ordering on edges. If we allow multiple preferences on worlds, the latter result can easily be generalised, thereby establishing that $\mathcal{L}^{\approx}$ and $\mathcal{L}^{\backsimeq}$ are equally expressive. This would have an interesting consequence, namely that the notions of 'normal effects' and 'normal executions' of actions are one and the same. This a priori counter-intuitive claim is easily justifiable. It turns out the effects of an action (the worlds one 'lands' in) depend to a large extent on what the current state of the world (the 'departing' points) is. In other terms, talking about effects (tacitly) amounts to talking about pairs (w, w'), linking both a context of execution and the action's outcome. This feature just carries over when normality is considered.

In this work, we have not addressed the question as to what an appropriate notion of entailment for $\mathcal{L}^{\approx}$ is and have contented ourselves with the standard (Tarskian) definition, which is monotonic (and therefore inappropriate in many contexts). The recent results by Booth *et al*. (2015) in a propositional setting may provide us with a springboard to investigate this matter in more expressive languages such as those we are interested in here.

Outlook on Further Work

We shall now briefly discuss about possible ideas for exploration stemming from the present work.

R-based Conditionals

A framework for representing and reasoning with defeasibility would not be complete without an account of (defeasible) conditionals. Here we catch a glimpse of two versions thereof which can both be defined in our *R*-ordered models semantics.

Given an *R*-ordered model $\mathscr{R}$, for every propositional sentence α, let $R_\alpha := \{(w, w') \mid \mathscr{R}, w \Vdash \alpha \text{ and } \mathscr{R}, w' \Vdash \alpha\}$ and $\ll_\alpha$ its corresponding preference relation. (Of course, if we work in a finite propositional language, then there are finitely many of such R_αs and $\ll_\alpha$s.) We can then define a conditional statement as a macro in $\mathcal{L}^{\approx}$ as follows:

- $\alpha \rightsquigarrow_1 \beta$ if and only if $\approx_\alpha \beta$.

Such a definition, of course, has its limitations, as it only allows for propositional sentences in the antecedent of the conditional. A generalisation to the case where $\alpha \in \mathcal{L}^{\approx}$ would hardly improve matters, as we would end up with an infinite number of accessibility relations in the *R*-component of our *R*-ordered models.

Fortunately, we can do better than this. First, we need to define an extra, identity relation *id* on *W* and order its elements in the same way as for the other *R*-components. The intuition of doing so is that the most normal *id*-arrows correspond to the most normal worlds, i.e., we get an ordering on worlds induced by the ordering on the elements of the identity relation. With this, we can define our second candidate for a conditional in the following way. First, for every $\alpha \in \mathcal{L}^{\approx}$, let $id^\alpha := \{(w, w) \in id \mid \mathscr{R}, w \Vdash \alpha\}$. Then

- $\mathscr{R} \Vdash \alpha \rightsquigarrow_2 \beta$ if and only if for every w such that $(w, w) \in \min_{\ll_{id}} id^\alpha$, it holds that $\mathscr{R}, w \Vdash \beta$.

We shall leave an investigation of the appropriateness of $\rightsquigarrow_2$ as a defeasible conditional for future work.

Next Steps in Preferential Reasoning for DLs

In the context of formal ontologies specified in Description Logics (Baader et al. 2007), placing a preference order on binary relations as we proposed here has a natural appeal. As an example, consider the role name $\mathsf{hasChild}$: 'Normal' tuples in this relation may be biological or adopted parent-child tuples, while an 'exceptional' tuple may be an appointed legal guardian parent-child tuple. In this example, there is nothing exceptional about either the legal guardian or the child—the exceptionality lies in the nature of their *relationship*.

To make things more precise, given a DL interpretation $\mathcal{I} = \langle \Delta^{\mathcal{I}}, \cdot^{\mathcal{I}} \rangle$, we can enrich it with a collection of preference relations $\ll^{\mathcal{I}} := \langle \ll^{\mathcal{I}}_{r_1}, \ldots, \ll^{\mathcal{I}}_{r_n} \rangle$, one for each role name and each of which satisfying the conditions in Definition 6. Armed with this semantic construction, it becomes possible to:

- Define *defeasible value restrictions* (Britz et al. 2013) of the form $\forall r.C$, like $\forall \mathsf{hasChild}.\mathsf{Male}$, which refers to those individuals whose most normal parenting relations are of male children;
- State *defeasible role inclusions* of the form $r_1 \mathrel{\underset{\sim}{\sqsubset}} r_2$, as in e.g. $\mathsf{parentOf} \mathrel{\underset{\sim}{\sqsubset}} \mathsf{progenitorOf}$, which stipulates that the role of being a parent is usually that of also being the progenitor;
- State *typicality-based role instances* in the ABox of the form $\bullet r(a, b)$, where $\bullet$ is the extension of a typicality operator (Booth, Meyer, and Varzinczak 2012; Giordano et al. 2007) to roles, like $\bullet\mathsf{hasChild}(\mathsf{john}, \mathsf{anne})$, conveying the information that, under the interpretation of role $\mathsf{hasChild}$, the tuple $(\mathsf{john}, \mathsf{anne})$ is to be regarded as a typical one;
- State *defeasible role properties* like in saying that role $\mathsf{marriedTo}$ is *normally functional* and that partOf is *normally transitive*, while allowing for exceptions, i.e., less normal tuples failing the relation's property under consideration.

Moreover, definitions analogous to those in the preceding subsection would allow us to:

- State *defeasible concept subsumptions* (Britz, Heidema, and Meyer 2008; Britz, Meyer, and Varzinczak 2011b; Casini and Straccia 2010; Giordano et al. 2007) of the form $C \mathrel{\underset{\sim}{\sqsubset}} D$, as in $\mathsf{Mother} \mathrel{\underset{\sim}{\sqsubset}} \exists\mathsf{marriedTo}$, of which the intuition is that usually, mothers are married.

It is an open question whether a result similar to that obtained in Theorem 1 holds in a DL context. Roles can be reified, similar to the reification of n-ary relations in DLs (Sattler, Calvanese, and Molitor 2007), as a workaround to model preferences on tuples as preferences on objects in a DL enriched with a preferential subsumption relation $\sqsubsetneq$. Nevertheless, it is not immediately clear how the addition of preferential roles to a DL with preferential subsumption would affect its expressivity.

Defeasible Comparative Epistemic Logic

By placing a preference relation on the accessibility relations, we can get to a generalisation of Comparative Epistemic Logic (CEL) (Ditmarsch, Hoek, and Kooi 2012).

In CEL, a statement of the form $a \geq b$ intuitively means "agent b knows at least as much as agent a". The corresponding semantics is given by:

- $\mathscr{M}, w \Vdash a \geq b$ if and only if $R_b(w) \subseteq R_a(w)$.

In the context of our enriched semantic framework, we could envisage making statements of the form "agent b *normally* knows as much as agent a", of which a semantics can be given by the condition $\min_{\ll_b} R_b^w \subseteq R_a^w$.

Summary and Conclusion

The contributions of the present paper can be summarised as follows: (*i*) the motivation for and the definition of a semantic structure allowing for the ordering of *pairs* of worlds (instead of worlds *tout court*, as is customary in traditional NMR formalisms) and (*ii*) a generalisation of bisimulation to the preferential case together with a result relating our new semantics to that we studied in previous work and showing that, in the *monomodal* case, they are equivalent.

We have introduced a logic allowing for modal operators the intuition of which is to capture the idea of some transitions being more normal than others. As we have seen, our R-ordered models can be used to provide the extended language with an intuitive and elegant semantics. The resulting framework provides for an alternative formalisation for the notion of defeasible necessity we studied previously.

We have given examples, in an action, epistemic and deontic contexts, of what this semantic structure, as simple as it is, would allow us to represent (or give a meaning to) that one cannot do with standard Kripkean semantics. Likewise, we have briefly illustrated the fruitfulness of our definitions in other formalisms, in particular in a DL setting.

Acknowledgements

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