A Contraction Core for Horn Belief Change: Preliminary Report

Richard Booth
University of Luxembourg
Luxembourg
richard.booth@uni.lu

Thomas Meyer
Meraka Institute, CSIR
South Africa
tommie.meyer@meraka.org.za

Ivan Varzinczak
Meraka Institute, CSIR
South Africa
ivan.varzinczak@meraka.org.za

Renata Wassermann
Universidade de São Paulo
Brazil
renata@ime.usp.br

Abstract

In this paper we continue recent investigations into belief change for Horn logic. The main contribution is a result which shows that the construction method for Horn contraction for belief sets based on infra-remainder sets, as recently proposed by Booth *et al.*, corresponds exactly to Hansson's classical kernel contraction for belief sets, when restricted to Horn logic. This result is obtained via a detour through Horn contraction for belief bases during which we prove that kernel contraction for Horn belief bases produces precisely the same results as the belief base version of the Booth *et al.* construction method. The use of belief bases to obtain the result provides evidence for the conjecture that Horn belief change is best viewed as a "hybrid" version of belief set change and belief base change. One of the consequences of the link with base contraction is the provision of a more elegant representation result for Horn contraction for belief sets in which a version of the Core-retainment postulate features. The paper focuses on Delgrande's entailment-based contraction (e-contraction), but we also mention similar results for inconsistency-based contraction (i-contraction) and package contraction (p-contraction).

Introduction

In his seminal paper, (Delgrande 2008) has shed some light on the theoretical underpinnings of belief change by weakening the usual assumption in the belief change community that the underlying logical formalism should be at least as strong as (full) classical propositional logic. Delgrande investigated *contraction* for *belief sets* (sets of sentences closed under logical consequence) for Horn clauses. Delgrande's main contributions were threefold. Firstly, he showed that the move to Horn logic leads to different types of contraction, referred to as *entailment-based contraction* and *inconsistency-based contraction*, which coincide in the full propositional case. Secondly, Delgrande showed that Horn contraction for belief sets does not satisfy the controversial *Recovery* postulate, but exhibits some characteristics that are usuallly associated with the contraction of *bases* (arbitrary sets of sentences). And finally, Delgrande made a tentative conjecture that a version of Horn contraction usually referrred to as *orderly maxichoice contraction* is the appropriate method for contraction in Horn theories.

In a subsequent paper, Booth et al. (2009) showed that while Delgrande's *partial meet* constructions are appropriate choices for contraction in Horn logic, they do not constitute *all* the appropriate forms of Horn contraction. They build on a more fine-grained construction for belief set contraction which we refer to in this paper as *infra contraction*. In addition to the two types of Horn contraction introduced by Delgrande, they also introduce a third.

However, the investigation into Horn belief contraction is not closed yet. For one thing, Booth *et al.*'s representation result has the rather unsatisfactory property that it relies on a postulate which refers directly to the construction method it is intended to characterise. Moreover, as referred to earlier, although Horn contraction is defined on Horn belief sets, it seems to be related in some ways to contraction for belief *bases*: an aspect which has not yet been explored properly.

In this paper we continue the investigation into contraction for Horn logic, and address both the issues mentioned above, as well as others. We bring into the picture a construction method for contraction first introduced by Hansson (1994), known as *kernel contraction*. Although kernel contraction is usually associated with base contraction, it can be applied to belief sets as well. Our main contribution is a result which shows that the infra contraction of Booth *et al.* corresponds exactly to a version of Hansson's kernel contraction for belief sets, when restricted to Horn logic. In order to prove this, we first take a close look at Horn contraction for belief bases, defining a base version of infra contraction and proving that this construction is equivalent to kernel contraction for Horn belief bases. Since Horn belief sets are not closed under classical consequence, they can be seen as a "hybrid" between belief sets and belief bases. This justifies the use of belief bases to obtain results for belief set Horn contraction.

The investigation into base contraction also affords us the opportunity to improve on the rather unsatisfactory representation result proved by Booth *et al.* for infra contraction. We show that a more elegant representation result can be obtained by replacing the postulate introduced by Booth *et al.* with the well-known Core-retainment postulate, which is usually associated with base contraction. The presence of Core-retainment here further enforces the hybrid nature of Horn belief change—lying somewhere between belief set change and base change.

The paper focuses on Delgrande's entailment-based contraction (e-contraction), but we also mention similar results for two other relevant types of Horn contraction.

Horn logic has found extensive use in Artificial Intelligence, in particular in logic programming, truth maintenance systems, and deductive databases.[1] This explains, in part, our interest in belief change for Horn logic. Another reason for focusing on this topic is because of its application to debugging and repairing ontologies in description logics (Baader et al. 2003). In particular, Horn logic can be seen as the backbone of the $\mathcal{EL}$ family of description logics (Baader, Brandt, and Lutz 2005), and a proper understanding of belief change for Horn logic is therefore important for finding solutions to similar problems expressible in the $\mathcal{EL}$ family.

The rest of the paper is structured as follows: The next section introduces the formal background needed. This is followed by discussions on base contraction and belief set contraction, after which we briefly review the existing work on propositional Horn contraction. The following three sections constitute the core of the paper. In the first one we prove that kernel contraction and infra contraction are equivalent on the level of bases. This enables us, in the following section, to prove that kernel contraction and infra contraction are equivalent on the Horn belief set level as well. And from this we are led in the next section to provide a more elegant characterisation for infra contraction for Horn belief sets than the one presented by Booth *et al.*. We wrap up with a section on related work which also concludes and discusses future directions of research.

Logical Background

We work in a finitely generated propositional language $\mathcal{L}_\mathsf{P}$ over a set of propositional atoms $\mathfrak{P}$, which includes the distinguished atoms $\top$ and $\bot$, and with the standard model-theoretic semantics. Atoms will be denoted by $p, q, \ldots$, possibly with subscripts. We use $\varphi, \psi, \ldots$ to denote classical formulas. They are recursively defined in the usual way.

Classical logical consequence and logical equivalence are denoted by $\models$ and $\equiv$ respectively. For $X \subseteq \mathcal{L}_\mathsf{P}$, the set of sentences logically entailed by X is denoted by $Cn(X)$. A *belief set* is a logically closed set, i.e., for a belief set K, $K = Cn(K)$. We usually denote belief sets by K, possibly decorated by primes. $\mathscr{P}(X)$ denotes the power set (set of all subsets) of X.

A *Horn clause* is a sentence of the form $p_1 \wedge p_2 \wedge \ldots \wedge p_n \rightarrow q$ where $n \geq 0$, $p_i, q \in \mathfrak{P}$ for $0 \leq i \leq n$ (recall that the $p_i s$ and q may be one of $\bot$ or $\top$ as well). If $n = 0$ we write q instead of $\rightarrow q$. A *Horn set* is a set of Horn clauses.

Given a propositional language $\mathcal{L}_\mathsf{P}$, the Horn language $\mathcal{L}_\mathsf{H}$ generated from $\mathcal{L}_\mathsf{P}$ is simply the Horn clauses occurring in $\mathcal{L}_\mathsf{P}$. The Horn logic obtained from $\mathcal{L}_\mathsf{H}$ has the same semantics as the propositional logic obtained from $\mathcal{L}_\mathsf{P}$, but just restricted to Horn clauses. A *Horn belief set*, usually denoted by H (possibly with primes), is a Horn set closed under logical consequence, but containing only Horn clauses. Hence, $\models$, $\equiv$, the $Cn(.)$ operator, and all other related notions are defined relative to the logic we are working in (e.g. $\models_\mathsf{PL}$ for propositional logic and $\models_\mathsf{HL}$ for Horn logic). We shall dispense with such subscripts where the context makes it clear which logic we are dealing with.

[1] Despite our interest in Horn clauses, it is worth noting that in this work we do not consider logic programming explicitly and we do not use negation as failure at all.

Base Contraction

Contraction is intended to represent situations in which an agent has to give up information, say a formula φ, from its current stock of beliefs. Other operations of interest in belief change are the *expansion* of an agent's current beliefs by φ, where the basic idea is to add φ regardless of the consequences, and the *revision* of its current beliefs by φ, where the intuition is to incorporate φ into the current beliefs in some way while ensuring consistency at the same time.

We commence with a discussion on *base contraction* where an agent's beliefs are represented as a set of sentences, also known as a *base*. We usually denote bases by B, possibly decorated with primes.

Definition 1 *A base contraction $-$ for a base B is a function from $\mathcal{L}_\mathsf{P}$ to $\mathscr{P}(\mathcal{L}_\mathsf{P})$.*

Intuitively the idea is that, for a fixed base B, contraction of a formula φ produces a new base $B - \varphi$.

One of the standard approaches for constructing belief contraction operators is based on the notion of a *remainder set*—a maximally consistent subset of B not entailing φ (Alchourrón, Gärdenfors, and Makinson 1985). Below we apply this to bases, but as we shall see, it can be applied to belief sets (closed under logical consequence) as well.

Definition 2 *Given a set B, $X \in B \bot \varphi$ iff (i) $X \subseteq B$; (ii) $X \not\models \varphi$; (iii) for all X' s.t. $X \subset X' \subseteq B$, $X' \models \varphi$. We call the elements of $B \bot \varphi$ the* remainder sets *of B w.r.t. φ.*

It is easy to verify that $B \bot \varphi = \emptyset$ if and only if $\models \varphi$.

Since there is no unique method for choosing between possibly different remainder sets, there is a presupposition of the existence of a suitable selection function for doing so.

Definition 3 *A* selection function *γ for a set B is a (partial) function from $\mathscr{P}(\mathscr{P}(\mathcal{L}_\mathsf{P}))$ to $\mathscr{P}(\mathscr{P}(\mathcal{L}_\mathsf{P}))$ such that $\gamma(B \bot \varphi) = \{B\}$ if $B \bot \varphi = \emptyset$, and $\emptyset \subset \gamma(B \bot \varphi) \subseteq B \bot \varphi$ otherwise.*

Selection functions provide a mechanism for identifying the remainder sets judged to be most appropriate, and the resulting contraction is then obtained by taking the intersection of the chosen remainder sets.

Definition 4 *For a selection function γ, the base contraction $-_\gamma$ generated by γ as follows: $B -_\gamma \varphi = \bigcap \gamma(B \bot \varphi)$ is a* base partial meet contraction.

Two subclasses of base partial meet deserve special mention.

Definition 5 *Given a selection function γ, $-_\gamma$ is a* base maxichoice *contraction iff $\gamma(B \bot \varphi)$ is always a singleton set. It is a* base full meet *contraction iff $\gamma(B \bot \varphi) = B \bot \varphi$ whenever $B \bot \varphi \neq \emptyset$.*

Base full meet contraction is unique, while base maxichoice contraction usually is not.

For reasons that will become clear, it is interesting to observe that the following *convexity principle* does not hold for belief bases.

(Convexity) For a base B, let $-_{mc}$ be a base maxichoice contraction, and let $-_{fm}$ be base full meet contraction. For every set X and φ s.t. $(B -_{fm} \varphi) \subseteq X \subseteq B -_{mc} \varphi$, there is a base partial meet contraction $-_{pm}$ s.t. $B -_{pm} \varphi = X$.

The principle simply states that every set between the results obtained from base full meet contraction and some base maxichoice contraction can be obtained from some base partial meet contraction. To see that it does *not* hold, let $B = \{p \to q, q \to r, p \wedge q \to r, p \wedge r \to q\}$ and consider contraction by $p \to r$. It is easily verified that base maxichoice gives either $B-(p \to r) = B' = \{p \to q, p\wedge r \to q\}$ or $B - (p \to r) = B'' = \{q \to r, p \wedge q \to r, p \wedge r \to q\}$ and therefore the only other result obtained from base partial meet contraction is that which is provided by base full meet contraction: $B - (p \to r) = B''' = \{p \wedge r \to q\}$. But observe that even though it is the case that $B''' \subseteq X \subseteq B''$ where $X = \{p \wedge q \to r, p \wedge r \to q\}$, X is not equal to any of B', B'', or B'''.

Base partial meet contraction can be characterised by the following postulates

$(B-1)$ If $\not\models \varphi$, then $B - \varphi \not\models \varphi$ (Success)

$(B-2)$ $B - \varphi \subseteq B$ (Inclusion)

$(B-3)$ If $B' \models \varphi$ if and only if $B' \models \psi$ for all $B' \subseteq B$, then $B - \varphi = B - \psi$ (Uniformity)

$(B-4)$ If $\psi \in (B \setminus (B - \varphi))$, then there is a B' s.t. $(B - \varphi) \subseteq B' \subseteq B$ and $B' \not\models \varphi$, but $B' \cup \{\psi\} \models \varphi$ (Relevance)

Theorem 1 (Hansson 1992) *Every base partial meet contraction satisfies* $(B-1)$–$(B-4)$*. Conversely, every base contraction which satisfies* $(B-1)$–$(B-4)$ *is a base partial meet contraction.*

Kernel contraction was introduced by Hansson (1994) as a generalization of *safe contraction* (Alchourrón and Makinson 1985). Instead of looking at maximal subsets not implying a given formula, kernel operations are based on minimal subsets that imply a given formula.

Definition 6 *For a base* B*,* $X \in B \perp\!\!\!\perp \varphi$ *iff (i)* $X \subseteq B$*; (ii)* $X \models \varphi$*; and (iii) for every* X' *s.t.* $X' \subset X$*,* $X' \not\models \varphi$*.* $B \perp\!\!\!\perp \varphi$ *is called the* kernel set *of* B *w.r.t.* φ *and the elements of* $B \perp\!\!\!\perp \varphi$ *are called the* φ-kernels of B.

The result of a base kernel contraction is obtained by removing at least one element from every (non-empty) φ-kernel of B, using an *incision function*.

Definition 7 *An* incision function σ *for a base* B *is a function from the set of kernel sets of* B *to* $\mathscr{P}(\mathcal{L}_{\mathsf{P}})$ *such that (i)* $\sigma(B \perp\!\!\!\perp \varphi) \subseteq \bigcup(B \perp\!\!\!\perp \varphi)$*; and (ii) if* $\emptyset \neq X \in B \perp\!\!\!\perp \varphi$*, then* $X \cap (\sigma(B \perp\!\!\!\perp \varphi)) \neq \emptyset$*.*

Definition 8 *Given an incision function* σ *for a base* B*, the* base kernel contraction $-_\sigma$ *for* B *generated by* σ *is defined as:* $B -_\sigma \varphi = B \setminus \sigma(B \perp\!\!\!\perp \varphi)$*.*

Base kernel contraction can be characterised by the same postulates as base partial meet contraction, except that Relevance is replaced by the Core-retainment postulate below:

$(B-5)$ If $\psi \in (B \setminus (B - \varphi))$, then there is some $B' \subseteq B$ such that $B' \not\models \varphi$ but $B' \cup \{\psi\} \models \varphi$ (Core-retainment)

Theorem 2 (Hansson 1994) *Every base kernel contraction satisfies* $(B-1)$–$(B-3)$ *and* $(B-5)$*. Conversely, every base contraction which satisfies* $(B-1)$–$(B-3)$ *and* $(B-5)$ *is a base kernel contraction.*

Observe that Core-retainment is slightly weaker than Relevance. And indeed, it thus follows that all base partial meet contractions are base kernel contractions, but as the following example from Hansson (1999) shows, some base kernel contractions are not base partial meet contractions.

Example 1 *Let* $B = \{p, p \vee q, p \leftrightarrow q\}$*. Then* $B \perp\!\!\!\perp (p \wedge q) = \{\{p, p \leftrightarrow q\}, \{p \vee q, p \leftrightarrow q\}\}$*, from which it follows that there is an incision function* σ *for* B *s.t.* $\sigma(B \perp\!\!\!\perp (p \wedge q)) = \{p \vee q, p \leftrightarrow q\}$*, and then* $B -_\sigma (p \wedge q) = \{p\}$*. On the other hand,* $B \perp (p \wedge q) = \{\{p, p \vee q\}, \{p \leftrightarrow q\}\}$*, from which it follows that base partial meet contraction* $B - (p \wedge q)$ *yields either* $\{p, p \vee q\}$*, or* $\{p \leftrightarrow q\}$*, or* $\{p, p \vee q\} \cap \{p \leftrightarrow q\} = \emptyset$*, none of which are equal to* $B -_\sigma (p \wedge q) = \{p\}$*.*

Belief Set Contraction

In *belief set* contraction the aim is to describe contraction on the *knowledge level* (Gärdenfors 1988) independent of how beliefs are represented. Thus, contraction is defined only for *belief sets* (i.e., sets closed under logical consequence).

Definition 9 *A* belief set contraction $-$ *for a belief set* K *is a function from* $\mathcal{L}_{\mathsf{P}}$ *to* $\mathscr{P}(\mathcal{L}_{\mathsf{P}})$*.*

By the principle of categorical matching (Gärdenfors and Rott 1995) the contraction of a belief set by a sentence is expected to yield a new belief set.

Given the construction methods for base contraction already discussed, two obvious first attempts to define belief set contraction are to consider both partial meet contraction and kernel contraction restricted to belief sets. For partial meet contraction this works well: the remainder sets (Definition 2) of belief sets are all belief sets as well, the definition of selection functions (Definition 3) carries over as is for belief sets, and partial meet contraction (Definition 4) when applied to belief sets will always yield a belief set as a result. When applied to belief sets, we refer to the contractions in Definition 4 as *belief set partial meet contractions*.

For kernel contraction matters are slightly more complicated. For one thing, for a belief set as input, base kernel contraction does not necessarily produce a belief set as a result. Of course, it is possible to ensure that a belief set is obtained by closing the result obtained from base kernel contraction under logical consequence.

Definition 10 *Given a belief set* K *and an incision function* σ *for* K*, the belief set contraction* $\approx_\sigma$ *for* K *generated as* *follows:* $K \approx_\sigma \varphi = Cn(K -_\sigma \varphi)$*, is a* belief set kernel contraction*.*

This is very closely related to a version of base contraction that Hansson (1999) refers to as *saturated base kernel contraction*:

Definition 11 *Given a base B and an incision function σ for B, the base contraction $\approx_\sigma$ for B generated as follows: $B \approx_\sigma \varphi = B \cap Cn(B -_\sigma \varphi)$, is a* saturated base kernel contraction*.*

It is easily shown that when the set B in the definition for saturated base kernel contraction is a belief set, the two notions coincide.

Observation 1 *For a belief set K and an incision function σ for K, the saturated base kernel contraction for σ and the belief set kernel contraction for σ are identical.*

While it can be shown that some saturated base kernel contractions are not base partial meet contractions, this distinction disappears when considering belief sets only.

Theorem 3 (Hansson 1994) *Let K be a belief set. A belief set contraction $-$ is a saturated base kernel contraction if and only if it is a belief set partial meet contraction.*

And as a result of Observation 1 and Theorem 3 we immediately have the following corollary:

Corollary 1 *Let K be a belief set. A belief set contraction $-$ is a belief set kernel contraction if and only if it is a belief set partial meet contraction.*

Belief set contraction defined in terms of partial meet contraction (and kernel contraction) corresponds exactly to what is perhaps the best-known approach to belief change: the so-called AGM approach (Alchourrón, Gärdenfors, and Makinson 1985). AGM requires that belief set contraction be characterised by the following set of postulates:

$(K-1)$ $K - \varphi = Cn(K - \varphi)$ (Closure)

$(K-2)$ $K - \varphi \subseteq K$ (Inclusion)

$(K-3)$ If $\varphi \notin K$, then $K - \varphi = K$ (Vacuity)

$(K-4)$ If $\not\models \varphi$, then $\varphi \notin K - \varphi$ (Success)

$(K-5)$ If $\varphi \equiv \psi$, then $K - \varphi = K - \psi$ (Extensionality)

$(K-6)$ If $\varphi \in K$, then $Cn((K - \varphi) \cup \{\varphi\}) = K$ (Recovery)

Alchourrø'n et al. (1985) have shown that these postulates characterise belief set partial meet contraction exactly.

Theorem 4 *Every belief set partial meet contraction satisfies $(K-1)$–$(K-6)$. Conversely, every belief set contraction which satisfies $(K-1)$–$(K-6)$ is a belief set partial meet contraction.*

We will not elaborate in detail on these postulates, except for a brief comparison with the postulates for base contraction. Closure is specific to belief set contraction and has no counterpart in base contraction. Both Inclusion and Success also occur as postulates for base contraction. A modified version of Vacuity (in which the antecedent is changed to "If $B \not\models \varphi$") holds for base contraction. Extensionality is a special case of the Uniformity postulate for base contraction. And finally, Recovery is stronger than both Relevance and Core-retainment, and does not hold for base contraction.

A final word on belief set contraction for full propositional logic: Booth et al. (2009) have shown that the Convexity property holds for belief sets.

Proposition 1 (Convexity) *Let K be a belief set, let $-_{mc}$ be a (belief set) maxichoice contraction, and let $-_{fm}$ be (belief set) full meet contraction. For every belief set X and sentence φ s.t. $(K -_{fm} \varphi) \subseteq X \subseteq K -_{mc} \varphi$, there is a (belief set) partial meet contraction $-_{pm}$ s.t. $K -_{pm} \varphi = X$.*

The result shows that every belief set between the results obtained from full meet contraction and some maxichoice contraction can also be obtained from some partial meet contraction. This means that it is possible in principle to define a version of belief set contraction based on such sets.

Definition 12 *For belief sets K and K', $K' \in K \downarrow \varphi$ if and only if there is some $K'' \in K \perp \varphi$ such that $(\bigcap K \perp \varphi) \subseteq K' \subseteq K''$. We refer to the elements of $K \downarrow \varphi$ as the* infra remainder sets *of K with respect to φ.*

Note that all remainder sets are also infra remainder sets, and so is the intersection of any set of remainder sets. Indeed, the intersection of any set of infra remainder sets is also an infra remainder set. So the set of infra remainder sets contains *all* belief sets between some remainder set and the intersection of all remainder sets. This explains why infra contraction below is not defined as the intersection of infra remainder sets (cf. Definition 4).

Definition 13 (Infra Contraction) *Let K be a belief set. An infra selection function τ is a (partial) function from $\mathscr{P}(\mathscr{P}(\mathcal{L}_\mathsf{P}))$ to $\mathscr{P}(\mathcal{L}_\mathsf{P})$ such that $\tau(K \downarrow \varphi) = K$ whenever $K \downarrow \varphi = \emptyset$, and $\tau(K \downarrow \varphi) \in K \downarrow \varphi$ otherwise. A belief set contraction $-_\tau$ is a* belief set infra contraction *if and only if $K -_\tau \varphi = \tau(K \downarrow \varphi)$.*

Because of Proposition 1 we can extend Corollary 1 to show that kernel contraction, partial meet contraction, and infra contraction all coincide for belief sets.

Corollary 2 *Let K be a belief set. A belief set contraction $-$ is a belief set kernel contraction iff it is a belief set partial meet contraction iff it is a belief set infra contraction.*

Horn Contraction

While there has been some work on *revision* for Horn clauses (Eiter and Gottlob 1992; Liberatore 2000; Langlois et al. 2008), it is only recently that attention has been paid to *contraction* for Horn logic. In particular, Delgrande (2008) investigated two distinct classes of contraction functions for Horn *belief sets*: e-contraction, for removing an unwanted consequence; and i-contraction, for removing formulas leading to inconsistency; while Booth et al. (2009) subsequently extended the work of Delgrande. Our focus in this paper is on e-contraction, although Delgrande, as well as Booth *et al.*, also consider i-contraction. Delgrande's definition of Horn logic allows for the *conjunction* of Horn clauses, and we shall follow his convention in this paper. (This differs from the definition of Booth *et al.* although their version can be recast into that of Delgrande without any loss—or gain—in expressivity.) Remembering, here we are going to use H, sometimes decorated with primes, to denote a Horn belief set.

Definition 14 *An* e-contraction $-$ *for a Horn belief set* H *is a function from* $\mathcal{L}_{\mathsf{H}}$ *to* $\mathscr{P}(\mathcal{L}_{\mathsf{H}})$.

Delgrande's method of construction for e-contraction is in terms of partial meet contraction. The definitions of remainder sets (Definition 2), selection functions (Definition 3), partial meet contraction (Definition 4), as well as maxichoice and full meet contraction (Definition 5) all carry over for e-contraction, with the set B in each case being replaced by a Horn *belief set* H, and we shall refer to these as e-remainder sets (denoted by $H\bot_e\varphi$), e-selection functions, partial meet e-contraction, maxichoice e-contraction and full meet e-contraction respectively (we leave out the reference to the term "Horn", since there is no room for ambiguity). As in the full propositional case, it is easy to verify that all e-remainder sets are also Horn belief sets, and that all partial meet e-contractions (and therefore the maxichoice e-contractions, as well as full meet e-contraction) produce Horn belief sets.

Although Delgrande defines and discusses partial meet e-contraction, he argues that maxichoice e-contraction (to be precise, a special case of maxichoice e-contraction referred to as *orderly* maxichoice e-contraction) is the appropriate approach for e-contraction. Booth et al. (2009), on the other hand, argue that although all partial meet e-contractions are appropriate choices for e-contraction, they do not make up the set of *all* appropriate e-contractions. The argument for appropriate e-contractions other than partial meet e-contraction is based on the observation that the convexity result for full propositional logic in Proposition 1 does not hold for Horn logic.

Example 2 (Booth, Meyer, and Varzinczak 2009) *Let* $H = Cn_{\mathsf{HL}}(\{p \rightarrow q, q \rightarrow r\})$. *It is easy to verify that, for the* e*-contraction of* $p \rightarrow r$*, maxichoice yields either* $H^1_{mc} = Cn_{\mathsf{HL}}(\{p \rightarrow q\})$ *or* $H^2_{mc} = Cn_{\mathsf{HL}}(\{q \rightarrow r, p \wedge r \rightarrow q\})$*, that full meet yields* $H_{fm} = Cn_{\mathsf{HL}}(\{p \wedge r \rightarrow q\})$*, and that these are the only three partial meet* e*-contractions. Now consider the Horn belief set* $H' = Cn_{\mathsf{HL}}(\{p \wedge q \rightarrow r, p \wedge r \rightarrow q\})$. *It is clear that* $H_{fm} \subseteq H' \subseteq H^2_{mc}$*, but there is no partial meet* e*-contraction yielding* H'.

In order to rectify this situation, Booth *et al.* propose that *every* Horn belief set between full meet and some maxichoice e-contraction ought to be seen as an appropriate candidate for e-contraction.

Definition 15 (Infra e-Remainder Sets) *For Horn belief sets* H *and* H', $H' \in H\downarrow_e\varphi$ *iff there is some* $H'' \in H\bot_e\varphi$ *s.t.* $(\bigcap H\bot_e\varphi) \subseteq H' \subseteq H''$. *We refer to the elements of* $H\downarrow_e\varphi$ *as the* infra e-remainder sets *of* H *w.r.t.* φ.

As with the case for full propositional logic, e-remainder sets are also infra e-remainder sets, and so is the intersection of any set of e-remainder sets. Similarly, the intersection of any set of infra e-remainder sets is also an infra e-remainder set, and the set of infra e-remainder sets contains *all* Horn belief sets between some e-remainder set and the intersection of all e-remainder sets. As in the full propositional case, this explains why e-contraction is not defined as the intersection of infra e-remainder sets (cf. Definition 4).

Definition 16 (Horn e-Contraction) *Let* H *be a Horn belief set. An infra e-selection function* τ *is a (partial) function from* $\mathscr{P}(\mathscr{P}(\mathcal{L}_{\mathsf{H}}))$ *to* $\mathscr{P}(\mathcal{L}_{\mathsf{H}})$ *such that* $\tau(H\downarrow_e\varphi) = H$ *whenever* $H\downarrow_e\varphi = \emptyset$*, and* $\tau(H\downarrow_e\varphi) \in H\downarrow_e\varphi$ *otherwise. An* e*-contraction* $-_\tau$ *is an infra* e*-contraction if and only if* $H -_\tau \varphi = \tau(H\downarrow_e\varphi)$.

Booth *et al.* show that infra e-contraction is captured precisely by the six AGM postulates for belief set contraction, except that Recovery is replaced by the following (weaker) postulate $(H -_e 6)$, and the Failure postulate (below) is added.

$(H -_e 6)$ If $\psi \in H \setminus (H - \varphi)$, then there exists an X s.t. $\bigcap(H\bot_e\varphi) \subseteq X \subseteq H$ and $X \not\models \varphi$, but $X \cup \{\psi\} \models \varphi$

$(H -_e 7)$ If $\models \varphi$, then $H -_e \varphi = H$ (Failure)

More formally (Booth, Meyer, and Varzinczak 2009):

Theorem 5 *Every infra e-contraction satisfies postulates* $(K-1)$–$(K-5)$, $(H-_e 6)$ *and* $(H-_e 7)$. *Conversely, every e-contraction which satisfies* $(K-1)$–$(K-5)$, $(H-_e 6)$ *and* $(H-_e 7)$ *is an infra e-contraction.*

Observe firstly that $(H -_e 6)$ bears some resemblance to the Relevance postulate for base contraction. Observe also that it is a somewhat unusual postulate in that it refers directly to e-remainder sets. It is possible to provide a more elegant characterisation of infra e-contraction as we shall see. Before we do so, however, we first take a detour through base contraction.

Base Infra Contraction

In the section on base contraction we have considered remainder sets for bases and kernel sets for bases, but not infra remainder sets for bases. We commence this section with the definition of *base infra remainders sets*.

Definition 17 (Base Infra Remainder Sets) *For bases* B *and* B', $B' \in B \downarrow \varphi$ *iff there is some* $B'' \in B\bot\varphi$ *s.t.* $(\bigcap B\bot\varphi) \subseteq B' \subseteq B''$. *We refer to the elements of* $B\downarrow\varphi$ *as the* base infra remainder sets *of* B *with respect to* φ.

Observe that the definition of base infra remainder sets is the same as for infra e-remainder sets, differing only in that (*i*) it deals with belief bases and not belief sets; and (*ii*) it is defined in terms of remainder sets for bases, and not for (Horn) belief sets.

Base infra remainder sets can clearly be used to define a form of base contraction in a way that is similar to that in Definitions 13 and 16.

Definition 18 (Base Infra Contraction) *A* base infra selection function τ *is a (partial) function from* $\mathscr{P}(\mathscr{P}(\mathcal{L}_{\mathsf{P}}))$ *to* $\mathscr{P}(\mathcal{L}_{\mathsf{P}})$ *s.t.* $\tau(B\downarrow\varphi) = B$ *whenever* $B\downarrow\varphi = \emptyset$*, and* $\tau(B\downarrow\varphi) \in B\downarrow\varphi$ *otherwise. A base contraction* $-_\tau$ *generated by* τ *as follows:* $B -_\tau \varphi = \tau(B\downarrow\varphi)$ *is a* base infra contraction.

A natural question to ask is how base infra contraction compares with base partial meet contraction and base kernel contraction. The following fundamental result, which plays a central role in this paper, shows that base infra contraction corresponds exactly to base kernel contraction.

Theorem 6 *A base contraction for a base B is a base kernel contraction for B iff it is a base infra contraction for B.*

From the section on base contraction, and from Example 1, specifically, we know that base kernel contraction is more general than base partial meet contraction—every base partial meet contraction is also a base kernel contraction, but the converse does not hold. From Theorem 6 it therefore follows that base infra contraction is more general than base partial meet contraction as well. This is not surprising, given that a similar result holds for partial meet e-contraction and infra e-contraction as we have seen in the section on Horn contraction (cf. Example 2).

Theorem 6 has a number of other interesting consequences as well. On a philosophical note, it provides corroborative evidence for the contention that the kernel contraction approach is more appropriate than the partial meet approach. The fact that kernel contraction is at least as general as partial meet contraction for both base and belief set contraction is already an argument favouring it over partial meet contraction. Theorem 6 adds to this by showing that a seemingly different approach to contraction (infra contraction), which is also at least as general as partial meet contraction for both base and belief set contraction, turns out to be identical to kernel contraction. As we shall see in the next section, Theorem 7 is also instrumental in "lifting" this result to the level of Horn belief sets.

Kernel e-contraction = Infra e-contraction

Through the work of Booth et al. (2009) we have already encountered partial meet contraction and infra contraction for Horn belief sets (partial meet e-contraction and infra e-contraction), but we have not yet defined a suitable version of kernel contraction for this case.

Definition 19 *Given a Horn belief set H and an incision function σ for H, the* Horn kernel e-contraction for *H, abbreviated as the* kernel e-contraction for *H is defined as $H \approx^e_\sigma \varphi = Cn_{\mathsf{HL}}(H -_\sigma \varphi)$, where $-_\sigma$ is the base kernel contraction for φ obtained from σ.*

Given the results on how kernel contraction, partial meet contraction and infra contraction compare for the base case (kernel contraction and infra contraction are identical, while both are more general than partial meet contraction), one would expect similar results to hold for Horn belief sets. And this is indeed the case. Firstly, infra e-contraction and kernel e-contraction coincide.

Theorem 7 *Given a Horn belief set H, an e-contraction for H is an infra e-contraction for H if and only if it is a kernel e-contraction for H.*

Proof sketch: Consider a base B and a formula φ. From Theorem 6 it follows that the set of base infra remainder sets of B w.r.t. φ (i.e., the set $B\downarrow\varphi$) is equal to the set of results obtained from the base kernel contraction of B by φ, call it KC^B_φ. Now let B be such that is a set of Horn clauses closed under Horn consequence (a Horn belief set) and φ a Horn clause. The elements of KC^B_φ are not necessarily closed under Horn consequence, but if we do close them, we obtain exactly the set of results obtained from kernel e-contraction when contracting B by φ (by the definition of kernel e-contraction). Let us refer to this latter set as $Cn_{\mathsf{HL}}(KC^B_\varphi)$. I.e., $Cn_{\mathsf{HL}}(KC^B_\varphi) = \{Cn_{\mathsf{HL}}(X) \mid X \in KC^B_\varphi\}$. Also, the elements of $B\downarrow\varphi$ are not closed under Horn consequence, but if we do close them the resulting set (refer to this set as $Cn_{\mathsf{HL}}(B\downarrow\varphi)$) contains exactly the infra e-remainder sets of B w.r.t. φ. I.e., $Cn_{\mathsf{HL}}(B\downarrow\varphi) = B\downarrow_e\varphi$ (why this is the case, will be explained below). But since $B\downarrow\varphi = KC^B_\varphi$, it is also the case that $Cn_{\mathsf{HL}}(B\downarrow\varphi) = Cn_{\mathsf{HL}}(KC^B_\varphi)$, and therefore $Cn_{\mathsf{HL}}(KC^B_\varphi) = B\downarrow_e\varphi$. [QED]

To see why $Cn_{\mathsf{HL}}(B\downarrow\varphi) = B\downarrow_e\varphi$, observe that since B is closed under Horn consequence, the (base) remainder sets of B w.r.t. φ (i.e., the elements of $B\perp\varphi$) are also closed under Horn consequence. So the elements of $B\downarrow\varphi$ are all the sets (not necessarily closed under Horn consequence) between $\bigcap(B\perp\varphi)$ and some element of $B\perp\varphi$. Therefore the elements of $Cn_{\mathsf{HL}}(B\downarrow\varphi)$ are all those elements of $B\downarrow\varphi$ that are closed under Horn consequence. That is, exactly the infra e-remainder sets of B with respect to φ.

From Theorem 7 and Example 2 it follows that partial meet e-contraction is more restrictive than kernel e-contraction. When it comes to Horn belief sets, we therefore have exactly the same pattern as we have for belief bases—kernel contraction and infra contraction coincide, while both are strictly more permissive than partial meet contraction. Contrast this with the case for belief sets for full propositional logic where infra contraction, partial meet contraction and kernel contraction all coincide.

One conclusion to be drawn from this is that the restriction to the Horn case produces a curious hybrid between belief sets and belief bases for full propositional logic. On the one hand, Horn contraction deals with sets that are logically closed. But on the other hand, the results for Horn logic obtained in terms of construction methods are close to those obtained for belief base contraction.

Either way, the new results on base contraction have proved to be quite useful in the investigation of contraction for Horn belief sets. In the next section we provide another result for Horn contraction inspired by the new results on base contraction in this paper.

An Elegant Characterisation for e-Contraction

In the section on Horn contraction we remarked that the characterisation of infra e-contraction, specifically the $(H -_e 6)$ postulate, is somewhat unusual in that it refers directly to an aspect of the construction method (e-remainder sets) that it is meant to characterise. In this section we show that it is possible to provide a more elegant characterisation of infra e-contraction—one that replaces $(H -_e 6)$ with the Core-retainment postulate which we encountered in the section on base contraction, and that is used in the characterisation of base kernel contraction.

Theorem 8 *Every infra e-contraction satisfies $(K-1)$-$(K-5)$ Core-retainment and $(H -_e 7)$. Conversely, every e-contraction which satisfies $(K-1)$-$(K-5)$, Core-retainment and $(H -_e 7)$ is an infra e-contraction.*

This result was inspired by Theorem 6 which shows that base kernel and base infra contraction coincide. Given that Core-retainment is used in characterising base kernel contraction, Theorem 6 shows that there is a link between Core-retainment and base infra contraction, and raises the question of whether there is a link between Core-retainment and infra e-contraction. The answer, as we have seen in Theorem 8, is yes. This result provides more evidence for the hybrid nature of contraction for Horn belief sets—in this case the connection with base contraction is strengthened.

Related Work and Concluding Remarks

We have mentioned the work on revision for Horn clauses by Eiter and Gottlob (1992), Liberatore (2000) and Langlois (2008). Apart from the work of Delgrande (2008) and Booth et al. (2009) which were discussed extensively in earlier sections, there is some recent work on obtaining a semantic characterisation of Horn contraction by Fotinopoulos (2009). Billington (1999) considered revision and contraction for defeasible logic which is quite different from Horn logic in many respects, but nevertheless has a rule-like flavour to it which has some similarity to Horn logic.

In bringing Hansson's kernel contraction into the picture, we have made useful and meaningful contributions to the investigation into contraction for Horn logic. More specifically, the main contributions of this paper are (*i*) a result which shows that infra contraction and kernel contraction for the base case coincide; (*ii*) lifting the previous results to Horn belief sets to show that infra contraction and kernel contraction for Horn belief sets coincide; and (*iii*) using these results as a guide to the provision of a more elegant characterisation of the representation result by Booth *et al.* for infra contraction as applied to Horn belief sets.

In addition to e-contraction, Delgrande investigated a version of Horn contraction he refers to as *inconsistency-based contraction* (or i-contraction) where the purpose is to modify an agent's Horn belief set in such a way as to avoid inconsistency when a sentence φ is provided as input. That is, an i-contraction $-_i$ should be such that $(H -_i \varphi) \cup \{\varphi\} \not\models_{HL} \bot$. In addition Booth *et al.* considered a version of *package contraction* (or p-contraction) by a set of sentences Φ, for which *none* of the sentences in Φ should be in the result obtained from p-contraction. Although it seems that the new results presented in this paper can be applied to both i-contraction and p-contraction, this still has to be verified in detail.

Finally, we have seen that kernel e-contraction and infra e-contraction are more general than partial meet e-contraction. But there is evidence that even these forms of Horn contraction may not be sufficient to obtain all meaningful answers, as can be seen from the following example.

Example 3 *Consider again our Horn belief set example $Cn_{\mathsf{HL}}(\{p \to q, q \to r\})$ encountered in Example 2. If we view basic Horn clauses (clauses with exactly one atom in the head and the body) as representative of arcs in a graph, in the style of the old inheritance networks, then one possible desirable outcome of a contraction by $p \to r$ is $Cn_{\mathsf{HL}}(q \to r)$. However, as we have seen in Example 2, this is not an outcome supported by infra e-contraction (and therefore not by kernel e-contraction either).*

Ideally, a truly comprehensive e-contraction approach for Horn logic would be able to account for such cases as well.

References

[1985] Alchourrón, C., and Makinson, D. 1985. On the logic of theory change: safe contraction. *Studia Logica* 44:405–422.

[1985] Alchourrón, C.; Gärdenfors, P.; and Makinson, D. 1985. On the logic of theory change: Partial meet contraction and revision functions. *J. of Symb. Logic* 50:510–530.

[2003] Baader, F.; Calvanese, D.; McGuinness, D.; Nardi, D.; and Patel-Schneider, P., eds. 2003. *Description Logic Handbook*. Cambridge University Press.

[2005] Baader, F.; Brandt, S.; and Lutz, C. 2005. Pushing the $\mathcal{EL}$ envelope. In *Proc. IJCAI*.

[1999] Billington, D.; Antoniou, G.; Governatori, G.; and Maher, M. 1999. Revising nonmonotonic theories: The case of defeasible logic. In *Proc. KI*.

[2009] Booth, R.; Meyer, T.; and Varzinczak, I. 2009. Next steps in propositional Horn contraction. In *Proc. IJCAI*.

[2008] Delgrande, J. 2008. Horn clause belief change: Contraction functions. In *Proc. KR*, 156–165.

[1992] Eiter, T., and Gottlob, G. 1992. On the complexity of propositional knowledge base revision, updates, and counterfactuals. *AI Journal* 57(2–3):227–270.

[2009] Fotinopoulos, A. M., and Papadopoulos, V. 2009. Semantics for Horn contraction. In *7th PanHellenic Logic Symposium*.

[1995] Gärdenfors, P., and Rott, H. 1995. Belief revision. In *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 4. Clarendon Press. 35–132.

[1988] Gärdenfors, P. 1988. *Knowledge in Flux*. MIT Press.

[1992] Hansson, S. O. 1992. A dyadic representation of belief. In Gärdenfors, P., ed., *Belief Revision*. Cambridge University Press. 89–121.

[1994] Hansson, S. 1994. Kernel contraction. *J. of Symbolic Logic* 59(3):845–859.

[1999] Hansson, S. O. 1999. *A Textbook of Belief Dynamics, Theory Change and Database Updating*. Dordrecht: Kluwer Academic Publishers.

[2008] Langlois, M.; Sloan, R.; Szörényi, B.; and Turán, G. 2008. Horn complements: Towards Horn-to-Horn belief revision. In *Proc. AAAI*.

[2000] Liberatore, P. 2000. Compilability and compact representations of revision of Horn clauses. *ACM Transactions on Computational Logic* 1(1):131—161.