Pertinent Reasoning

Katarina Britz
Meraka Institute and UNISA
Pretoria, South Africa
arina.britz@meraka.org.za

Johannes Heidema
University of South Africa
Pretoria, South Africa
johannes.heidema@gmail.com

Ivan Varzinczak
Meraka Institute, CSIR
Pretoria, South Africa
ivan.varzinczak@meraka.org.za

Abstract

In this paper we venture beyond one of the fundamental assumptions in the non-monotonic reasoning community, namely that non-monotonic entailment is *supra*-classical. We investigate reasoning which uses an infra-classical entailment relation that we call *pertinent entailment*. The notion of pertinence proposed here is induced by a binary accessibility relation on worlds establishing a link (representing some form of pertinence) between premiss and consequence. We show that this notion can be captured elegantly using a simple modal logic without nested modalities. One road to infra-classicality has been studied extensively, that of *substructural* logics, which weaken the generating engine of axioms and inference rules for producing entailment pairs (X, Y). Here we follow an alternative strategy: we first demand that X entails Y classically, and then, with supplementary information provided by an accessibility relation, more, trimming down the set of entailment pairs to infra-classicality. It turns out that pertinent entailment restricts well-known 'paradoxes' avoided by relevance/relevant logic in an interesting way. We present its properties, showing that it possesses other non-classical properties, like strong non-explosiveness and non-monotonicity, and we discuss which inference rules traditionally considered in the literature it satisfies.

Introduction

Classical logic is, in a sense, the logic of complete ignorance. In a classical entailment $X \models Y$ no information whatsoever — beyond that encapsulated locally in X and Y — plays any role at all. Extra information may be employed to construct altered entailment relations, which sometimes allow *more* pairs (X, Y) into the relation, going *supra-classical*, or *fewer*, going *infra-classical*, or just going *non-classical*.

If rather specific, the extra information is usually expressed as syntactic rules or is of a semantic nature and typically involves an (often binary) relation on *W*, the set of 'worlds'. More generally and vaguely the 'extra' may be a desire to adapt classical entailment $\models$ in order to obtain an entailment relation which more closely resembles commonsense human reasoning as precipitated in natural language.

Pertinent reasoning is quite specific. It is based on infra-classical entailment relations which employ the information present in a binary accessibility relation *R* on *W* in so far as this is embodied in standard modal operators $\Box$ and $\Diamond$ with their *R*- and R^{-} (converse of *R*)-semantics. The information in *R* is considered to be *pertinent* to the sense in which X entails Y. This will be motivated in more detail and illustrated by means of examples.

In this work we consider *infra-classical* pertinence relations. One road to infra-classicality is well known, that of *substructural* logics (Restall 2006), which weaken the generating engine of *axioms* and *inference rules* for producing entailment pairs (X, Y). In pertinent reasoning we follow, in a sense, the opposite strategy: we first demand that $X \models Y$, but then (invoking *R*) *more*, trimming down the set of entailment pairs to infra-classicality.

The present text is structured as follows: after some logical preliminaries, we motivate and define infra-classical pertinent entailment. Following that, we investigate the non-classical properties satisfied by our entailment relation. We then derive a set of inference rules for pertinent reasoning. After a discussion of and comparison with related work, we conclude with an overview and future directions of research.

Logical Background

We work in a propositional language $\mathcal{L}$ over a set of propositional atoms $\mathfrak{P}$, together with the two distinguished atoms $\top$ (*verum*) and $\bot$ (*falsum*), and with the standard model-theoretic semantics. Atoms will be denoted by $p, q, \ldots$ We use $X, Y, \ldots$ to denote classical (Boolean) formulas. They are recursively defined in the usual way, with connectives $\neg$, $\wedge$, $\vee$, $\rightarrow$ and $\leftrightarrow$.

We denote by *W* the set of all *worlds* (alias propositional valuations or interpretations) $w : \mathfrak{P} \longrightarrow \{0, 1\}$, with 0 denoting falsity and 1 truth. Satisfaction of X by w is denoted by $w \Vdash X$. With $Mod(X)$ we denote the set of all models of X (propositional valuations satisfying X).

Classical logical consequence (semantic entailment) and logical equivalence are denoted by $\models$ and $\equiv$ respectively. Given sentences X and Y, the meta-statement $X \models Y$ means $Mod(X) \subseteq Mod(Y)$. $X \equiv Y$ is an abbreviation (in the meta-language) of $X \models Y$ *and* $Y \models X$.

We now extend our propositional language with one modal operator $\Box$ (Blackburn, van Benthem, and Wolter 2006). We will denote complex formulas (possibly with modal operators) by $\Phi, \Psi, \ldots$ They are recursively defined as follows:

$$\Phi ::= X \mid \Box X \mid \neg\Phi \mid \Phi \wedge \Phi \mid \Phi \vee \Phi \mid \Phi \to \Phi \mid \Phi \leftrightarrow \Phi$$

With $\mathfrak{F}$ we denote the set of all complex formulas of our language. Note that no nesting of modal operators is allowed. Our only reason for this restriction is expositional simplicity. Our work extends naturally to a fully (multi-) modal context. Here we will also use the modal operator $\Diamond$, which is the *dual* operator of $\Box$, defined by $\Diamond X \equiv_{\text{def}} \neg\Box\neg X$.

Definition 1 *A* frame *is a tuple $\mathscr{F} = \langle W, R\rangle$, with W the set of* worlds *and $R \subseteq W \times W$ the* accessibility relation *on W.*

For simplicity of exposition our notion of frame does not follow the standard notion from modal logics: here no two worlds satisfy the same valuation. Nevertheless, all we shall say in the sequel can be straightforwardly formulated for standard frames.

Sometimes it will be useful to consider the *identity relation* on W. It is defined as $id_W := \{(w, w) \mid w \in W\}$. For purposes that will become clear in the sequel, in this paper we consider only *reflexive* frames, i.e., we assume that $id_W \subseteq R$.

Definition 2 *Given a frame $\mathscr{F} = \langle W, R\rangle$,*

- $w \Vdash^{\mathscr{F}} p$ *(p is true at world w of frame $\mathscr{F}$) iff $w \Vdash p$;*
- $w \Vdash^{\mathscr{F}} \Box X$ *iff $w' \Vdash^{\mathscr{F}} X$ for every w' such that $(w, w') \in R$;*
- $w \Vdash^{\mathscr{F}} \neg\Phi$ *iff $w \nVdash^{\mathscr{F}} \Phi$, i.e., if it is not the case that $w \Vdash^{\mathscr{F}} \Phi$;*
- $w \Vdash^{\mathscr{F}} \Phi \wedge \Psi$ *iff $w \Vdash^{\mathscr{F}} \Phi$ and $w \Vdash^{\mathscr{F}} \Psi$;*
- *truth conditions for the other connectives are classical.*

Given a frame $\mathscr{F} = \langle W, R\rangle$ and formulas Φ, Ψ, we say that Φ *entails* Ψ with respect to frame $\mathscr{F}$ (denoted $\Phi \models^{\mathscr{F}} \Psi$) if and only if for every $w \in W$, if $w \Vdash^{\mathscr{F}} \Phi$, then $w \Vdash^{\mathscr{F}} \Psi$. If $\top \models^{\mathscr{F}} \Phi$, we say that Φ is (logically) *valid* (or a *tautology*) in frame $\mathscr{F}$ and we denote this as $\models^{\mathscr{F}} \Phi$. Clearly, for X and Y both without modal operators, $X \models^{\mathscr{F}} Y$ is equivalent to $X \models Y$ (X entails Y classically).

Given modal operators $\Box$ and $\Diamond$, we can speak of their *converse* operators: $\Box^-$ and $\Diamond^-$, respectively. The following definition follows straightforwardly from Definition 2 by applying the converse of the accessibility relation R, but since we are going to refer constantly to these notions we state them here:

Definition 3 *Given a frame $\mathscr{F} = \langle W, R\rangle$,*

- $w' \Vdash^{\mathscr{F}} \Box^- X$ *iff $w \Vdash^{\mathscr{F}} X$ for every w such that $(w, w') \in R$;*
- $w' \Vdash^{\mathscr{F}} \Diamond^- X$ *iff $w \Vdash^{\mathscr{F}} X$ for some w such that $(w, w') \in R$.*

Finally, we have another useful definition:

Definition 4 *Let $\mathscr{F} = \langle W, R\rangle$. For any $U \subseteq W$:*

- $R[U] := \{w' \in W \mid$ *there is a $w \in U$ s.t.* $(w, w') \in R\}$;
- $R^-[U] := \{w \in W \mid$ *there is a $w' \in U$ s.t.* $(w, w') \in R\}$.

Hence, given a frame $\mathscr{F} = \langle W, R\rangle$ and a formula X, it is easy to see that e.g.

$Mod(\Diamond X) = R^-[Mod(X)]$ and $Mod(\Diamond^- X) = R[Mod(X)]$.

The Road to Pertinence

Classical semantic entailment $X \models Y$ says that $Mod(X) \subseteq Mod(Y)$, i.e., that every X-world is a Y-world. This formal definition does of course *not* capture all of the intuitive connotations of natural language phrases like "if X, then Y", "X entails Y", or "from X, Y follows logically". Many of the properties of $\models$ that may strike some people as 'odd' result from the following fact: As long as every X-world is a Y-world, $X \models Y$ and hence (equivalently) $Y \equiv X \vee (Y \wedge \neg X)$ hold, and the Y-worlds which are *not* in $Mod(X)$ are completely free and arbitrary, in the sense that they need have *nothing whatsoever* to do with X or any of the X-worlds. Any arbitrary ('trivial') dilation of $Mod(X)$ yields a Y such that $X \models Y$. One intuitive connotation of 'entailment' is that more, some additional relation of 'relevance' or 'pertinence', should hold between X and Y.

Existing relevance/relevant logics (Anderson and Belnap 1975; Anderson, Belnap, and Dunn 1992) share some of the aims that we have with the present paper, but (at least in our view) they harbour certain less attractive features:

Remark 1 Most of the literature on relevance/relevant logics confuse and conflate entailment with the conditional connective or 'material implication' ($\to$), the first being a notion at the *meta*-level and the second at the *object* level. According to Anderson and Belnap, "it is philosophically respectable to 'confuse' implication or entailment with the conditional, and indeed philosophically suspect to harp on the danger of such a 'confusion'" (Anderson and Belnap 1975, p. 473).

Remark 2 Relevance logics traditionally tend to start out from *syntactic* considerations to rule out some classical entailments as irrelevant and then afterwards contrive to constructing a matching semantics — not always completely convincingly, it must be said. Syntax is protean (shapeshifting): infinitely many syntactically different sentences represent the same proposition. Granted: there are normal forms. But our contention is that we should start from *semantic* notions and then find apt syntax to simulate the semantics.

Remark 3 Sometimes philosophical, metaphysical ideas get admixed into the relevance endeavour — ideas like 'dialetheism' (the thesis that some contradictions are 'true') or belief in 'impossible worlds', like 'inconsistent models of arithmetic' (Priest 2002). These notions may bemire an already complex issue.

Remark 4 Relevance logics traditionally pay scant attention to *contexts*. What is relevant in one reasoning context may not be so in another context. For instance, legal argument differs from intuitionistic proof in mathematics.

How our approach deals with these four issues will become clear in the sequel.

All of this is not to say, of course, that relevance/relevant logics are not appropriate candidates for pertinent reasoning. Here we follow an alternative (not antagonistic) approach. This is what we develop in the rest of the paper.

A Pertinent Entailment Relation

The notion of entailment is an asymmetric, directed relation. In the 'forward' (from premiss to consequence) direction it preserves truth, or at least plausibility; in the 'backward' direction it carries along falsity, or at least implausibility. In the forward direction, it usually loses information, while in the backward direction it usually gains information (think of hypothesis generation or abduction, for example).

In a direct proof of an entailment there is a step-by-step 'logical movement' from premiss to consequence; in an indirect proof, such as *reductio ad absurdum* or by contraposition, from the negation of the consequence to the negation of the premiss — 'directed movement', to and from.

This intuitive notion of entailment as a species of access relation between sentences or propositions — starting at the premiss *access to* the consequence, or starting at the consequence *access from* the premiss — this idea of entailment as 'access' has a natural analogue in the *accessibility relation* between *worlds* in modal logic. We intend to anchor our brand of 'relevant' entailment, called *pertinent* entailment, in some accessibility relation on worlds. To be specific: in the entailment relation which we choose as the focus of this paper, those totally unconstrained Y-worlds which are *not* X-worlds in a classical entailment $X \models Y$, should be disciplined. They should be admitted only if they have some pertinence to the premiss X — a pertinence that those Y-worlds which *are* X-worlds of course automatically have.

In our new infra-classical entailment of Y by X, the condition that we impose upon the (previously wild) $Y \wedge \neg X$-worlds is that now each of them must be *accessible* from *some* X-world. This establishes the mutual pertinence of X and Y to each other. Of course, this assumes that the specific accessibility relation chosen for this purpose reflects the required type of pertinence. (See below for more on the definition of *R* and examples.)

Given a propositional language, a frame $\mathscr{F} = \langle W, R \rangle$, with *R* a reflexive (and for some results in the sequel transitive) accessibility relation on the set of possible worlds *W*; and a modal operator $\Diamond^-$, corresponding to the converse relation R^- of *R*:

Definition 5 $X \mathrel{|\!\!<} Y$ *if and only if* $X \models^{\mathscr{F}} Y$ *and* $Y \models^{\mathscr{F}} \Diamond^- X$.

Intuitively, Definition 5 states that premiss X and consequence Y are *mutually pertinent* if and only if X entails Y (classically) and every Y-world in frame $\mathscr{F}$ is accessible from *some* X-world — importantly, the $Y \wedge \neg X$-worlds (Figure 1). (The X-worlds are each accessible from itself.)

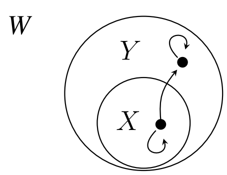

Figure 1: Pertinent entailment of Y by X: *Mod*$(X) \subseteq$ *Mod*(Y) and *any* Y-world is accessible from *some* X-world.

In the symbol $|\!\!<$, the '$<$' refers to infra-classicality, as opposed to the '$=$' in $\models$. We note that $|\!\!<$ can be defined equivalently, but more concisely and elegantly:

Proposition 1 $X \mathrel{|\!\!<} Y$ *if and only if* $X \vee Y \equiv Y \wedge \Diamond^- X$.

Given a premiss X, the set of consequences that X entails in our new relation are all the Ys that lie between that particular X-premiss and $\Diamond^- X$, and hence form a sub-lattice (closed under conjunction and disjunction) of the classical Lindenbaum-Tarski algebra of the language as depicted in Figure 2 (with classical entailment going 'up').

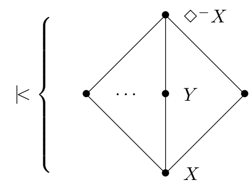

Figure 2: The sub-lattice induced by pertinent entailment from premiss X.

Given a consequence Y, the set of all those premisses X such that $X \mathrel{|\!\!<} Y$ does not always constitute a sub-lattice of the Lindenbaum-Tarski algebra, since it is not, in general, closed under conjunction. But it is closed under disjunction: if $X_1 \mathrel{|\!\!<} Y$ and $X_2 \mathrel{|\!\!<} Y$, then $X_1 \vee X_2 \mathrel{|\!\!<} Y$.

The second part in Definition 5 adds 'pertinence' to the classical entailment. It says: at every Y-world we can look back to some world, possibly different from where we are, and from which we could have come, in which X is true. The pertinence resides in the fairly subtle relationship required between (*i*) the truth values of sentences, and (*ii*) the accessibility between worlds. Obviously, $|\!\!<$ is an *infra-classical* entailment relation: if $X \mathrel{|\!\!<} Y$, then $X \models Y$.

Put in another way, we can easily see that our extra condition is equivalent to saying that $\Box^- \neg X \models^{\mathscr{F}} \neg Y$. ($\Box^-$ is the dual operator of $\Diamond^-$, defined by $\Box^- X \equiv_{\text{def}} \neg \Diamond^- \neg X$.) Now *Mod*$(\Box^- \neg X)$ is the set of all those $\neg X$-worlds w' such that every w from which w' can be accessed (wRw') is also a $\neg X$-world. Let us call this set of worlds the $\neg X$-*stream* within the set of all $\neg X$-worlds. Then the condition $\Box^- \neg X \models^{\mathscr{F}} \neg Y$ can be expressed metaphorically as follows:

> "The $\neg X$-*stream* flows within the set of $\neg Y$-worlds and completely misses *Mod*(Y)" (Figure 3).

The only general restriction that we put on *R* is that it must be reflexive: $id_W \subseteq R \subseteq W \times W$. The *minimum* (with respect to $\subseteq$) case, $R = id_W$, corresponds to the *maximum pertinence* of the relation $|\!\!<$, namely the case $|\!\!<\, = \,\equiv$ (i.e., classical logical equivalence, since now $Y \models^{\mathscr{F}} \Diamond^- X$ says that $Y \models X$). On the other hand, let $\models_<$ denote $\models \setminus \{(\bot, Y) \mid Y \not\equiv \bot\}$. Then the *maximum* case $R = W \times W$ corresponds to the *minimum pertinence* of $|\!\!<$, namely when $|\!\!<\, = \,\models_<$ (since now $Y \models^{\mathscr{F}} \Diamond^- X$ says that $Y \not\equiv \bot$ implies $X \not\equiv \bot$). Therefore we have

Theorem 1 *If R is reflexive, then* $\equiv \subseteq |\!\!< \subseteq \models_<$.

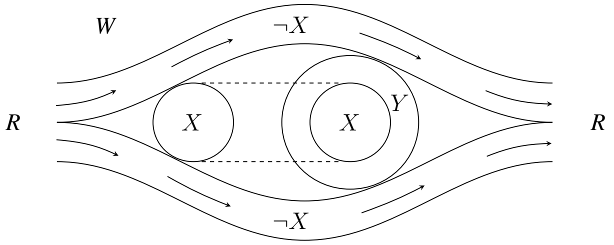

Figure 3: The 'rock' of X-worlds shields the 'island' of Y-worlds from the R-flow of the $\neg X$-stream.

Furthermore, given R_1 and R_2, if $R_1 \subseteq R_2$, then $|\!<_1 \subseteq |\!<_2$, where $|\!<_i$, $i \in \{1, 2\}$, is the pertinent entailment associated with the respective R_i.

Example 1 Let p be interpreted as the statement "Mars orbits the Sun", and q as the statement "a red teapot is orbiting Mars", and let $K = \{p\}$ be a set of background assumptions. Then K gives rise to an accessibility relation R, obtained by removing from $W \times W$ all links from $\neg p$-worlds to p-worlds (Figure 4).

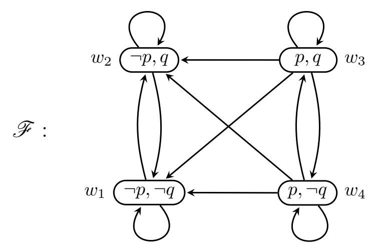

Figure 4: R induced by background assumptions $K = \{p\}$.

The intuition behind any R as defined in this example is that it restricts entailments from premisses that conflict with K. In particular, no entailment from premiss X, with X conflicting with K, to conclusion Y, with Y consistent with K, is allowed.

When the premiss is consistent with K, the entailment coincides with classical entailment. For example, the following entailments are valid: $p \wedge q \;|\!<\; p$; $p \wedge q \;|\!<\; q$; $p \;|\!<\; p \vee q$; $q \;|\!<\; p \vee q$; $p \;|\!<\; \top$; and $q \;|\!<\; \top$. However, when the premiss *contradicts* the background assumptions, the entailment relation is restricted: if X contradicts K, and $X \;|\!<\; Y$, then Y also contradicts K. This extends the strong non-explosiveness of $|\!<$ to the sterility of premisses contradicting background assumptions. For example, none of the following entailments are valid: $\neg p \wedge q \;|\!<\; q$; $\neg p \;|\!<\; \neg p \vee q$; and $\neg p \;|\!<\; \top$.

Properties of Pertinent Entailments

Now we discuss some of the properties of our pertinent entailment relation $|\!<$. We have already seen that $|\!<$ is the entailment $\models_<$ in the special case when $R = W \times W$, and $\equiv$ when $R = id_W$.

Pertinent entailment $|\!<$ is *non-explosive* in the strong sense that *falsum* is not omnigenerating, in fact, only self-generating: if $\bot \;|\!<\; Y$, then $Y \equiv \bot$. No contingent or tautological proposition is $|\!<$-entailed by a contradiction. This follows from the fact that for $\bot \;|\!<\; Y$ to hold, $Y \models^{\mathscr{F}} \Diamond^- \bot$ has to be the case, which holds only when $Y \equiv \bot$. Note that this weak form of paraconsistency does not involve any metaphysical ideas (cf. Remark 3), and that all contradictions are logically equivalent.

Our pertinent entailment relation is *paratrivial* in the sense that *verum* is not omnigenerated, but only from premisses with very special properties. Consider $X \;|\!<\; \top$ with an X which is not tautologous. Then, given a frame $\mathscr{F} = \langle W, R \rangle$, we have $W = Mod(\top) = Mod(X) \cup Mod(\neg X)$. From $X \;|\!<\; \top$, we get $\top \models^{\mathscr{F}} \Diamond^- X$, and then it follows that every world, and in particular every $w' \in Mod(\neg X)$, is accessible from some $w \in Mod(X)$, wRw' — indeed a rather strong stricture on X (and R). Intuitively, the assumption that from the X-worlds collectively *every* world whatsoever can be accessed justifies the mutual pertinence of X and $\top$.

Classical *disjunctive syllogism* — $(\neg X \vee Y) \wedge X \models Y$, which is equivalent to $Y \wedge X \models Y$ — is a minor pet hate of some relevance and relevant logicians: "the disjunctive syllogism is the only conceivable problematic rule of inference [amongst those under consideration]", (Dunn and Restall 2002, p. 33). Even though classically we have no problem with $(\neg X \vee Y) \wedge X \models Y$, one can appreciate that in $Y \wedge X \models Y$ the X is rather irrelevant. Does $|\!<$ help to isolate some 'relevant' (pertinent) cases of disjunctive syllogism?

Given a frame $\mathscr{F} = \langle W, R \rangle$, $Y \wedge X \;|\!<\; Y$ means that $Y \wedge X \models^{\mathscr{F}} Y$ and $Y \models^{\mathscr{F}} \Diamond^-(Y \wedge X)$. We then have $Mod(Y) \subseteq R[Mod(Y \wedge X)]$: to every Y-world one can come from some (possibly other) $Y \wedge X$-world. Every Y-world, even if not an X-world, can be reached from some $Y \wedge X$-world. This establishes the pertinence of Y and X to each other. (This also means that if Y is consistent, then so is $Y \wedge X$ — but that we already know from the strong non-explosiveness of $|\!<$.)

Classical $(\neg X \vee Y) \wedge X \models Y$ is a version of *modus ponens*, viz. the *resolution rule* — while there are at least four different versions of *modus ponens* (Cantwell 2009, p. 51). For our $|\!<$ we then saw that it holds only in a restricted and controlled way. A general form of *modus ponens* for $|\!<$ is proved in the next section.

Another interesting property of $|\!<$ is that, irrespective of the choice of R, the set of non-modal *pertinent tautologies* of the modal propositional language is identical to the set of non-modal classical tautologies:

Theorem 2 *Given any reflexive frame $\mathscr{F}$, $\top \;|\!<\; Y$ if and only if $\top \models Y$.*

All tautologies, irrespective of their syntactical form, are semantically equivalent — and trivial — having all of W as their models. This means that they do not exclude any possibility and contain no semantic information whatsoever. From a classical semantic point of view there is no justification for accepting some tautologies (say $X \vee \neg X$) but then rejecting others (say $X \rightarrow (Y \rightarrow X)$). Syntactically different sentence forms without any difference of model sets, of semantic meaning, are not treated differently and can be substituted anytime and anywhere by each other in our semantic approach. All the tautologies together are just one undifferentiated element $\top$ in the Lindenbaum-Tarski algebra of propositions, or, if you like, logical equivalence classes of sentences. Relevance/pertinence makes only sense relative to some extra semantic information (whether reflected on the object/syntactic level in a sentence or available only on the semantic/meta-level), while undifferentiated *W* has none. Pertinent entailment needs to move out of the domain of triviality, of tautologies, of "huh? — we know nothing!"

Classically, we have *contraposition*: $X \models Y$ is equivalent to $\neg Y \models \neg X$. Not so for $|\!<$, and proof by contradiction does not hold in general. $\neg Y \,|\!<\, \neg X$ says that $X \models Y$ and $\neg X \models^{\mathscr{F}} \Diamond^{-}\neg Y$: every X-world *is* a Y-world and every $\neg X$-world *can be reached from* some $\neg Y$-world. This may be an infra-classical entailment worthy of study.

Now one question that naturally arises is whether the classical meta-theorem called *deduction*, or by some authors the *Ramsey test* for conditionals (Cantwell 2009) ($X \models Y$ is equivalent to $\top \models X \to Y$), also holds for $|\!<$. So, is it the case that $X \,|\!<\, Y$ if and only if $\top \,|\!<\, X \to Y$?

For the left-to-right direction, suppose that $X \,|\!<\, Y$, i.e., for a given frame $\mathscr{F} = \langle W, R \rangle$, $X \models^{\mathscr{F}} Y$ and $Y \models^{\mathscr{F}} \Diamond^{-}X$. Then $\top \models^{\mathscr{F}} X \to Y$ and surely $X \to Y \models^{\mathscr{F}} \Diamond^{-}\top$ (since the accessibility relation *R* has been assumed to be reflexive).

Now, for the right-to-left direction, let us suppose that $\top \,|\!<\, X \to Y$, i.e., $\top \models^{\mathscr{F}} X \to Y$ and $X \to Y \models^{\mathscr{F}} \Diamond^{-}\top$. The second statement is just the triviality $X \to Y \models^{\mathscr{F}} \top$, i.e., $X \to Y \models \top$. We do not (in general) get the needed $Y \models^{\mathscr{F}} \Diamond^{-}X$.

Hence, $X \,|\!<\, Y$ implies $\top \,|\!<\, X \to Y$, but not conversely — unless every Y-world is accessible from some X-world, which is precisely the pertinence and the infra-classicality of $|\!<$.

We noted in Theorem 2 above that the sets of classical and of pertinent (non-modal) tautologies are identical. While classical entailment $X \models Y$ is equivalent to $X \to Y$ being a tautology, this is false for pertinent entailment. For the latter, "to harp on the danger" of conflating entailment and conditional is indeed pertinently *not* "philosophically suspect" (remember Remark 1).

In our approach it is not difficult to define a modal conditional connective which *does* satisfy the Ramsey test. We define the modal binary connective $\diamond\!\!\rightarrow$, called the *pertinent conditional* as follows:

Definition 6 $X \diamond\!\!\rightarrow Y \equiv_{def} (X \to Y) \wedge (Y \to \Diamond^{-}X)$.

Theorem 3 $X \,|\!<\, Y$ *if and only if* $|\!<\, X \diamond\!\!\rightarrow Y$.

One of the specific *bêtes noires* of relevance and relevant logicians is what they call *positive paradox* and write as $\alpha \to (\beta \to \alpha)$. With the introduction of our (stricter) conditional $\diamond\!\!\rightarrow$, one question that naturally arises is whether we have a pertinent version of positive paradox. The answer, as expected, is 'no', as shown by the following result:

Proposition 2 $\not|\!<\, \alpha \diamond\!\!\rightarrow (\beta \diamond\!\!\rightarrow \alpha)$.

Corollary 1 $\alpha \not|\!<\, \beta \diamond\!\!\rightarrow \alpha$.

We finish this section by observing that pertinent entailment $|\!<$ also satisfies substitution of semantic equivalents: Let $\alpha \,|\!<\, \beta$ and γ be a subformula of α. Then, for every γ' such that $\models \gamma \leftrightarrow \gamma'$, we have that $\alpha' \,|\!<\, \beta$, where α' is obtained by uniformly substituting γ' for γ in α. Consequently, we do not have the *variable sharing property* required in relevance logics (Dunn and Restall 2002).

Rules of Pertinent Inference

With regards to a proof theory for $|\!<$, i.e., a sound and complete syntactical counterpart for our semantic entailment, we can resort to existing decision procedures, notably tableaux (Goré 1999) and resolution (de Nivelle, Schmidt, and Hustadt 2000), for both conditions in Definition 5. We do not develop this further here; however we do investigate which desirable inference rules hold for $|\!<$. These rules are studied in the literature on non-monotonic reasoning and in particular supra-classical preferential reasoning (Kraus, Lehmann, and Magidor 1990; Lehmann and Magidor 1992; Britz, Heidema, and Labuschagne 2009).

Modus Ponens Pertinent entailment $|\!<$ satisfies the rule *modus ponens* (or disjunctive syllogism, cf. previous discussion) in the following sense:

$$\frac{X \,|\!<\, Y,\ X \,|\!<\, Y \to Z}{X \,|\!<\, Z}$$

Reflexivity $|\!<$ also satisfies the rule of *reflexivity*:

$$X \,|\!<\, X$$

From the assumed reflexivity of the accessibility relation *R*, reflexivity of $|\!<$ follows immediately.

And The following *and* rule is satisfied by $|\!<$:

$$\frac{X \,|\!<\, Y,\ X \,|\!<\, Z}{X \,|\!<\, Y \wedge Z}$$

This rule can indeed be strengthened by weakening any one of the two premisses to $X \models Y$ or $X \models Z$. (And, when introducing Figure 2, we already mentioned that the set of pertinent consequences of a fixed premiss X constitutes a sub-lattice of the Lindenbaum-Tarski algebra of the language.)

Or Pertinent entailment $|\!<$ satisfies the *or* rule below:

$$\frac{X \,|\!<\, Z,\ Y \,|\!<\, Z}{X \vee Y \,|\!<\, Z}$$

The proof also follows straightforwardly from our definitions; following Figure 2, we already mentioned that the set of premisses which pertinently entail the same Z is closed under disjunction. Again, the rule can be strengthened by weakening any one of the two premisses to $X \models Z$ or $Y \models Z$.

Non-Monotonicity Pertinent entailment $|\!<$ is *non-monotonic*, that is, the following monotonicity rule *fails*:

$$\frac{X \,|\!<\, Y,\ Z \models X}{Z \,|\!<\, Y}$$

So, assuming $X \mathrel{|\!\!<} Y$, we have *no* guarantee that $X \wedge X' \mathrel{|\!\!<} Y$: some Y-world may not be accessible from *any* $X \wedge X'$-world, even though it is accessible from some X-world. (Remember that we have already discussed *disjunctive syllogism*, where we saw that $Y \wedge X \mathrel{|\!\!<} Y$ holds only in very special pertinent cases.)

This result stands in contrast to one of the fundamental assumptions in the non-monotonic reasoning literature, viz. that non-monotonic entailment relations are *a priori* supra-classical. On the other hand, our pertinent entailment relations does satisfies the following weaker version of monotonicity:

Cautious Monotonicity (Cautious Left Strengthening)

$$\frac{X \models Y,\ X \mathrel{|\!\!<} Z}{X \wedge Y \mathrel{|\!\!<} Z}$$

Moreover, the following rules are also satisfied:

Cut (Cautious Left Weakening)

$$\frac{X \wedge Y \mathrel{|\!\!<} Z,\ X \models Y}{X \mathrel{|\!\!<} Z}$$

Cautious Right Weakening

$$\frac{X \mathrel{|\!\!<} Y,\ Z \models Y}{X \mathrel{|\!\!<} Y \vee Z}$$

Now let us assume that the accessibility relation inducing pertinence is not only reflexive but also *transitive*. Therefore we consider frames that are reflexive and transitive. This yields a pertinent entailment that satisfies some additional, and contextually desirable rules, notably:

Pertinent Transitivity (Pertinent Left Strengthening)

$$\frac{X \mathrel{|\!\!<} Y,\ Y \mathrel{|\!\!<} Z}{X \mathrel{|\!\!<} Z}$$

Furthermore, if *R* is transitive, then the consequence operator $Cn : \mathscr{P}(\mathfrak{F}) \longrightarrow \mathscr{P}(\mathfrak{F})$ corresponding to $\mathrel{|\!\!<}$ and defined by $Cn(\mathcal{Z}) := \{Y \mid X \mathrel{|\!\!<} Y \text{ for some } X \in \mathcal{Z}\}$ is a *closure operator*.

Related Work

Substructural, Relevance and Relevant Logics

Most of the existing work on relevance/relevant and substructural logics have mainly focused on the syntax: weakening the generating engine of *axioms* and *inference rules* to get rid of unwanted entailments (Anderson and Belnap 1975; Anderson, Belnap, and Dunn 1992; Dunn and Restall 2002; Restall 2006). Other existing approaches are algebraic in nature (Galatos et al. 2007). For those reasons, the referred works are not directly comparable to ours, since for present purposes we follow a *semantic*-driven approach to pertinence.

Quite recently, after an early start (Urquhart 1972), a few publications concentrated on further developing a proper semantics for relevant logics (Meyer 2004). There, a possible worlds approach is also defined, but with the aid of a *ternary* accessibility relation between worlds. Having (w, w', w'') in the accessibility relation means different things for different authors. For Meyer "[w]orlds are best demythologized as theories", and then, paraphrasing, theory w'' consists of all the outputs got by applying modus ponens in a certain way to major premisses from w and minor premisses from w'. Here we propose a simpler approach, viz. via a *binary* accessibility relation on plain *W*, for carrying out the required construction of pertinence.

Meyer showed that in almost all standard relevance logics the *relevant* conditional (usually written as $\rightarrow$) *cannot* be defined as a modalised truth function (Meyer 1975). More explicitly, Meyer proved that no standard *relevant* conditional can be represented as a "*strict*" $\Box\phi(\alpha, \beta)$, where $\phi(\alpha, \beta)$ is any truth functional combination of α and β. This prescription that "modalizing" in this context must mean "having $\Box$ as main connective" is of course restrictive. Our modal treatment in Definition 6 and Theorem 3 of the *pertinent* conditional connective $\diamond\!\!\rightarrow$ shows that by lifting pertinence to the meta-level we can achieve the desired result in a still quite elegant way, even if not with the main operator $\Box$.

Research on substructural logic usually adopts a *bottom-up* strategy (going from 'nothing' up to the entailments considered as relevant), and quite often via a proof-theoretic approach. Here we have followed a semantic-based *top-down* strategy: we start from classical logic and then go down to infra-classicality by culling the impertinent entailments. A similar strategy is that of 'filtering out' undesirable classical entailment pairs to prevent 'explosion' (*ex contradictione quodlibet*) in some paraconsistent logics (Priest 2002, pp. 297–299).

Supra-classical Logics

In Artificial Intelligence, there has been a great deal of work done on non-monotonic consequence relations (Kraus, Lehmann, and Magidor 1990; Boutilier 1994; Makinson 2005). As we saw in the section on rules of pertinent inference, our non-monotonic inference relation shares a lot of the properties that are viewed as important in that setting. A crucial difference between our work and the aforementioned is that our pertinent entailment is *infra*-classical, and therefore applicable in different contexts.

Much more in-depth research remains to be done in relating our work to existing notions of non-monotonic inference. However for present purposes we suffice with an observation contrasting our pertinent conditional (Definition 6) with Cantwell's defeasible conditional (Cantwell 2009):

Cantwell also argues (with many examples) for an indispensable distinction between his $X \mathrel{|\!\!\sim} Y$ (from the *assumption* or supposition X the consequence Y has to be accepted) and his $\mathrel{|\!\!\sim} X \rightarrow Y$ (irrespective of any assumption, the conditional $X \rightarrow Y$ *is accepted*). But the context and nature of his $\mathrel{|\!\!\sim}$ and $\rightarrow$ — with the purpose of discriminating strictly between *supposing* and *accepting* a statement — lead to properties very different from those of our pertinent entailment relation $\mathrel{|\!\!<}$: in most interesting cases Cantwell's valid statement $X \mathrel{|\!\!\sim} Y$ does *not* imply a valid $\mathrel{|\!\!\sim} X \rightarrow Y$. Both his $\mathrel{|\!\!\sim}$ and his $\rightarrow$ are non-classical, while our $\rightarrow$ is strictly classical (cf. Definition 6 for a non-classical version thereof); and his $\mid\!\sim$ is supra-classical.

Concluding Remarks

In this paper we have laid the groundwork for a semantic approach to the notion of pertinence. We have done that by investigating an infra-classical entailment relation that we call *pertinent entailment*, which turns out to be non-monotonic. In doing so, we have challenged one of the fundamental assumptions in the non-monotonic reasoning community, viz. that non-monotonic entailment is *supra*-classical, being derived by weakening (i.e., expanding) the underlying monotonic classical entailment relation. Arieli and Avron also question this assumption, but their framework still assumes that non-monotonic reasoning is derived by *weakening* some underlying monotonic entailment relation, e.g. paraconsistent entailment (Arieli and Avron 2000), whereas our pertinent entailment is obtained by *strengthening* monotonic classical entailment. Our purpose is also different from that usually assumed in the non-monotonic reasoning community, namely capturing *pertinence* (with additional information) rather than *plausibility* (with defeasible information).

We have shown that our notion of pertinence can be captured elegantly using a simple modal logic without nested modalities. We have seen (Theorem 1) that our flexible semantic approach allows for a whole spectrum of pertinent entailments, ranging between $\equiv$ and $\models_<$, and offering potential reasoning tools for many different contexts (cf. Remark 4). We are currently investigating an extension of our framework to a full (multi-) modal setting, in which notions such as *obligations* and *causality* can be captured.

Although our accessibility relation is different from preference relations studied in the context of defeasible reasoning (Hansson 2001), there are notable similarities worth investigating, as indicated by the inference rules satisfied by pertinent reasoning.

In this work we have investigated one case of infra-classicality, which is in the spirit of substructural logics like relevance and relevant logics. We plan to pursue future work by investigating further cases as well as the *supra-classical* counterparts of our entailment relations, which relate to prototypical reasoning and other forms of venturous reasoning.

References

[Anderson and Belnap 1975] Anderson, A., and Belnap, N. 1975. *Entailment: the logic of relevance and necessity*, volume 1. Princeton University Press.

[Anderson, Belnap, and Dunn 1992] Anderson, A.; Belnap, N.; and Dunn, J. 1992. *Entailment: the logic of relevance and necessity*, volume 2. Princeton University Press.

[Arieli and Avron 2000] Arieli, O., and Avron, A. 2000. General patterns for nonmonotonic reasoning: From basic entailments to plausible relations. *Logic Journal of the IGPL* 8:119–148.

[Blackburn, van Benthem, and Wolter 2006] Blackburn, P.; van Benthem, J.; and Wolter, F. 2006. *Handbook of Modal Logic*. Elsevier North-Holland.

[Boutilier 1994] Boutilier, C. 1994. Conditional logics of normality: A modal approach. *Artificial Intelligence* 68(1):87–154.

[Britz, Heidema, and Labuschagne 2009] Britz, K.; Heidema, J.; and Labuschagne, W. 2009. Semantics for dual preferential entailment. *Journal of Philosophical Logic* 38:433–446.

[Cantwell 2009] Cantwell, J. 2009. Conditionals in reasoning. *Synthese* 171:47–75.

[de Nivelle, Schmidt, and Hustadt 2000] de Nivelle, H.; Schmidt, R.; and Hustadt, U. 2000. Resolution-based methods for modal logics. *Logic Journal of the IGPL* 8(3):265–292.

[Dunn and Restall 2002] Dunn, J., and Restall, G. 2002. Relevance logic. In Gabbay, D., and Günthner, F., eds., *Handbook of Philosophical Logic*, volume 6. Kluwer Academic Publishers, 2nd edition. 1–128.

[Galatos et al. 2007] Galatos, N.; Jipsen, P.; Kowalski, T.; and Ono, H. 2007. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Studies in Logic and the Foundations of Mathematics. Elsevier.

[Goré 1999] Goré, R. 1999. Tableau methods for modal and temporal logics. In D'Agostino, M.; Gabbay, D.; Hähnle, R.; and Posegga, J., eds., *Handbook of Tableau Methods*. Kluwer Academic Publishers. 297–396.

[Hansson 2001] Hansson, S. 2001. *The Structure of Values and Norms*. Cambridge Studies in Probability, Induction and Decision Theory. Cambridge University Press.

[Kraus, Lehmann, and Magidor 1990] Kraus, S.; Lehmann, D.; and Magidor, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artificial Intelligence* 44:167–207.

[Lehmann and Magidor 1992] Lehmann, D., and Magidor, M. 1992. What does a conditional knowledge base entail? *Artificial Intelligence* 55:1–60.

[Makinson 2005] Makinson, D. 2005. How to go nonmonotonic. In *Handbook of Philosophical Logic*, volume 12. Springer, 2nd edition. 175–278.

[Meyer 1975] Meyer, R. 1975. Relevance is not reducible to modality. In Anderson, A., and Belnap, N., eds., *Entailment: the logic of relevance and necessity*, volume 1. Princeton University Press. 462–471.

[Meyer 2004] Meyer, R. 2004. Ternary relations and relevant semantics. *Annals of Pure and Applied Logic* 127:195–217.

[Priest 2002] Priest, G. 2002. Paraconsistent logic. In Gabbay, D., and Günthner, F., eds., *Handbook of Philosophical Logic*, volume 6. Kluwer Academic Publishers, 2nd edition. 287–393.

[Restall 2006] Restall, G. 2006. Relevant and substructural logics. In Gabbay, D., and Woods, J., eds., *Handbook of the History of Logic*, volume 7: Logic and the Modalities in the Twentieth Century. Elsevier North-Holland. 289–398.

[Urquhart 1972] Urquhart, A. 1972. Semantics for relevant logics. *Journal of Symbolic Logic* 37:159–169.