Pertinence Construed Modally

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Abstract

Capturing the notion of pertinence or relevance in logic is usually attempted at the *meta*-level. It can be induced either by specific *extra* information, or by general philosophical principles. In this paper we pay attention to both these origins. We present a semantic modal interpretation of the idea that there are two distinct relationships between a premiss and a conclusion that are pertinent to each other: of *semantic entailment* in the forward direction from α to β, and of *semantic constraint* in the backward direction from β to α. Unpacking the notion of pertinence into these two semantic components yields a class of entailment relations with appealing properties.
We define an entailment relation via a modal logic, and investigate its behaviour as a viable candidate for capturing the notion of pertinence. This approach allows us to deal with a number of paradoxes of material and strict implication (e.g. positive paradox), as well as some counter-intuitive properties of classical (and modal) entailment (e.g. explosiveness and disjunctive syllogism), in a satisfactory way. Furthermore, the resulting logic is infra-modal, non-monotonic, and allows for non-trivial reasoning with inconsistencies.

Keywords: Pertinence, modal logic KT, infra-modal entailment, non-explosiveness, non-monotonicity

1 Introduction

Classical logic is, to some extent, the logic of complete ignorance. Given a classical entailment $\alpha \models \beta$, no information whatsoever — beyond that encapsulated locally in α and β — plays any role at all. Extra information may be employed to construct altered entailment relations, which sometimes allow *more* pairs (α, β) into the relation, going *supra-classical*, or *fewer*, going *infra-classical*, or just going *non-classical*.

Classical semantic entailment $\alpha \models \beta$ says that every α-world is a β-world. This formal definition does of course *not* capture all of the intuitive connotations of natural language phrases like "if α, then β", "α entails β", or "from α, β follows logically". Many of the properties of $\models$ that may strike some people as 'odd' result from the following fact: As long as every α-world is a β-world, $\alpha \models \beta$, and hence (equivalently) $\beta \equiv \alpha \vee (\beta \wedge \neg\alpha)$, hold, and the β-worlds which do *not* satisfy α are completely free and arbitrary, in the sense that they need have *nothing whatsoever* to do with α or any of the α-worlds.

Any arbitrary ('trivial') dilation of the set of (classical) valuations satisfying α yields a β such that $\alpha \models \beta$. One intuitive connotation of 'entailment' is that more, some additional relation of 'relevance' or 'pertinence', should hold between α and β.

If rather specific, such extra information is usually expressed either as syntactic rules or as semantic constraints, and typically involves an (often binary) relation on the set of sentences of the language. More generally and vaguely the 'extra' may be a desire to adapt classical entailment $\models$ in order to obtain an entailment relation which more closely resembles human reasoning as precipitated in natural language.

In this paper we follow a semantic rather than syntactic approach, and consider pertinence relations which can be seen as *infra-modal* in the following sense: Similar to infra-classical entailments which are obtained by trimming classical Boolean entailment, we obtain infra-modal entailments by trimming standard modal entailments.

One road to infra-classicality is well known, that of *substructural* logics [25], which weaken the generating engine of *axioms* and *inference rules* for producing entailment pairs (α, β). In pertinent reasoning we follow, in a sense, the opposite strategy: we first demand that $\alpha \models \beta$, but then (invoking extra information in the meta-level) *more*, trimming down the set of entailment pairs to infra-modal consequence.

Existing relevance and relevant logics [1] [1,2] share some of the aims that we have with the present paper. However they harbour certain less attractive features. See Avron's critique [4] of early relevance logics for more details on Remarks 1.1–1.3 below.

Remark 1.1 Much of the literature on relevance and relevant logics confuse and conflate entailment with the conditional connective or 'material implication' $(\rightarrow)$, the first being a notion at the *meta*-level and the second at the *object* level. According to Anderson and Belnap, "it is philosophically respectable to 'confuse' implication or entailment with the conditional, and indeed philosophically suspect to harp on the danger of such a 'confusion'" [1, p. 473].

Remark 1.2 Relevance logics traditionally tend to start out from *syntactic* considerations to rule out some classical entailments as irrelevant and then afterwards contrive to constructing a matching semantics [1,2]. Syntax is protean (shape-shifting): Infinitely many syntactically different sentences represent the same proposition. Granted — there are normal forms. But our contention is that we should start from *semantic* notions and then find apt syntax to simulate the semantics.

Remark 1.3 Interpreting the conjunction and disjunction connectives in a purely truth-functional way leads to a number of counter-intuitive results; by definition, these connectives are not sensitive to intensional considerations, which includes both specific extra semantic information and general philosophical principles of relevance.

Remark 1.4 Sometimes philosophical, metaphysical ideas get admixed into the relevance endeavour — ideas like 'dialetheism' (the thesis that some contradic-

[1] There seems to be no agreement in the literature on how to denote the logics of relevance. Here we take a neutral position and speak of both *relevance* and *relevant logics*, as well as *relevance* and *relevant logicians*.

tions are 'true') or belief in 'impossible worlds', like 'inconsistent models of arithmetic' [24]. These notions may bemire an already complex issue.

Remark 1.5 Relevance logics traditionally pay scant attention to *contexts*. What is relevant in one reasoning context may not be so in another context. For instance, legal argument differs from intuitionistic proof in mathematics.

How our approach deals with these issues will become clear in the sequel. All of this is not to say, of course, that relevance and relevant logics are not appropriate candidates for pertinent reasoning. Here we follow an alternative (not antagonistic) approach.

The remainder of this paper is organized as follows: after some logical preliminaries (Section 2), we motivate and define a modal pertinent entailment relation (Section 3). In Section 4, we analyze the properties satisfied by our pertinent entailment. We then discuss the adequacy of our constructions as a candidate for capturing the notion of relevance by showing how they deal with some of the 'paradoxes' usually avoided by relevant logicians. In Section 5 we give examples illustrating both the intuitiveness and versatility of our definitions. Finally, after a discussion of and comparison with related work (Section 6), we conclude with an overview and future directions of investigation.

2 Logical Preliminaries

We work in a propositional (and at least for now mono-) modal language $\mathcal{L}$ over a set of propositional atoms $\mathfrak{P}$, together with the distinguished atom $\top$ (*verum*), and with the normal modal operator $\Box$ [8,13]. Atoms are denoted by $p, q, \ldots$ The formulas of our modal language are denoted by $\alpha, \beta, \ldots$ Those are recursively defined as follows:

$$\alpha \ ::= p \mid \top \mid \neg\alpha \mid \alpha \wedge \alpha \mid \Box\alpha$$

All the other connectives and the special atom $\bot$ (*falsum*) are defined in terms of $\neg$ and $\wedge$ in the usual way. As expected, the dual of $\Box$, namely $\Diamond$, is defined by $\Diamond\alpha \equiv_{\text{def}} \neg\Box\neg\alpha$.

Definition 2.1 A *model* is a tuple $\mathscr{M} = \langle W, R, V \rangle$, where

- W is a set of *worlds*;
- $R \subseteq W \times W$ is an *accessibility relation* on W; and
- $V : \mathfrak{P} \times W \longrightarrow \{0, 1\}$ is a *valuation*.

Definition 2.2 Given a model $\mathscr{M} = \langle W, R, V \rangle$,

- $w \Vdash^{\mathscr{M}} p$ if and only if $V(p, w) = 1$;
- $w \Vdash^{\mathscr{M}} \top$ for every $w \in W$;
- $w \Vdash^{\mathscr{M}} \neg\alpha$ if and only if $w \nVdash^{\mathscr{M}} \alpha$;
- $w \Vdash^{\mathscr{M}} \alpha \wedge \beta$ if and only if $w \Vdash^{\mathscr{M}} \alpha$ and $w \Vdash^{\mathscr{M}} \beta$;
- $w \Vdash^{\mathscr{M}} \Box\alpha$ if and only if $w' \Vdash^{\mathscr{M}} \alpha$ for every w' such that $(w, w') \in R$;
- truth conditions for the other connectives are as usual.

Definition 2.3 Given a model $\mathscr{M} = \langle W, R, V \rangle$ and a formula α,

- If $w \Vdash^{\mathscr{M}} \alpha$ for a given $w \in W$, we say that w *satisfies* α, or *is a model of* α with respect to $\mathscr{M}$;
- If $w \Vdash^{\mathscr{M}} \alpha$ for every $w \in W$, we say that α is *valid* in $\mathscr{M}$, noted $\models^{\mathscr{M}} \alpha$.

In the present paper we are interested in the class of models having a *reflexive* accessibility relation, i.e., given $\mathscr{M} = \langle W, R, V \rangle$, $id_W \subseteq R$, where id_W is the *identity relation* on W. (The reasons why we restrict ourselves to reflexive models will be made clear in the sequel.) This defines the modal logic KT [13]. (Nevertheless, all the above definitions remain the same, just restricted now to the class of KT-models.)

In this paper we employ the following versions of *local consequence*:

Definition 2.4 Given a KT-model $\mathscr{M} = \langle W, R, V \rangle$ and formulas α and β, we say that α *entails* β *in* $\mathscr{M}$ (noted $\alpha \models^{\mathscr{M}} \beta$) if and only if for every $w \in W$, if $w \Vdash^{\mathscr{M}} \alpha$, then $w \Vdash^{\mathscr{M}} \beta$.

Definition 2.5 Given a class $\mathscr{C}$ of KT-models and formulas α, and β,

- If $\alpha \models^{\mathscr{M}} \beta$ for every $\mathscr{M} \in \mathscr{C}$, we say that α *entails* β *in* $\mathscr{C}$ (noted $\alpha \models^{\mathscr{C}} \beta$);
- If $\models^{\mathscr{M}} \alpha$ for every $\mathscr{M} \in \mathscr{C}$, we say that α is *valid* in $\mathscr{C}$ (noted $\models^{\mathscr{C}} \alpha$);
- If $\neg\alpha$ is *not* valid in $\mathscr{C}$, we say that α is *satisfiable* in $\mathscr{C}$.

A specific class of models can be determined by imposing additional axiom schemas (e.g. transitivity, reflexivity, etc.) or by means of *global axioms* (formulas one wants to be valid in the class) [7,17]. Examples 5.1 and 5.2 later on will illustrate ways in which a class of KT-models $\mathscr{C}$ may be defined.

Since the class of models we are working with will be made clear from the context, for the sake of readability we shall dispense with superscripts and just write $\alpha \models \beta$ instead of $\alpha \models^{\mathscr{C}} \beta$.

3 Modal Pertinent Entailment

The notion of entailment is an asymmetric, directed relation. In the 'forward' (from premiss to consequence) direction it preserves truth, or at least plausibility; in the 'backward' direction it carries along falsity, or at least implausibility. In the forward direction, it usually loses information, while in the backward direction it usually gains information (think of hypothesis generation or abduction, for example).

In a direct proof of an entailment there is a step-by-step 'logical movement' from premiss to consequence; in an indirect proof, such as *reductio ad absurdum* or by contraposition, from the negation of the consequence to the negation of the premiss — 'directed movement', to and from.

This intuitive notion of entailment as a species of access relation between sentences or propositions — starting at the premiss *access to* the consequence, or starting at the consequence *access from* the premiss — this idea of entailment as 'access' has a natural analogue in the *accessibility relation* between *worlds* in modal logic. On the other hand, it has been known for quite a while that precisely notions such as relevance *cannot* be captured by standard modalities, by accessibility relations on *worlds* [22]: Relevance is a relation between *sentences* (sets of worlds), and not (at least in principle) between worlds alone.

Nevertheless, we intend to anchor our *pertinent* entailment in some notion in the meta-level *making use* of the accessibility relation on worlds. To be specific: in the entailment relation which we choose as the focus of this paper, those totally unconstrained β-worlds which are *not* α-worlds in a modal entailment $\alpha \models \beta$, should be disciplined. They should be admitted only if they have some pertinence to the premiss α — a pertinence that those β-worlds which *are* α-worlds of course automatically have.

In our new entailment of β by α, the condition that we impose upon the (previously wild) $\beta \wedge \neg\alpha$-worlds is that now each of them must be *accessible* from *some* α-world. This establishes the mutual pertinence of α and β to each other. But note that this is not to say that the pertinence is between *worlds*. It is rather between the *sets* of α- and β-worlds. Of course, this assumes that the specific subclass of accessibility relations chosen for this purpose reflects the required type of pertinence. See below for more on the definition of a class of models and examples.

Given our normal modal operators $\Box$ and $\Diamond$, we can speak of their *converse* operators, $\breve{\Box}$ and $\breve{\Diamond}$, respectively. The following definition follows straightforwardly from Definition 2.2 by applying the converse $\breve{R}$ of the accessibility relation R, but since we are going to refer constantly to these notions we state them here:

Definition 3.1 Given a model $\mathscr{M} = \langle W, R, V\rangle$,

- $w' \Vdash^{\mathscr{M}} \breve{\Box}\alpha$ if and only if $w \Vdash^{\mathscr{M}} \alpha$ for every w such that $(w, w') \in R$;
- $w' \Vdash^{\mathscr{M}} \breve{\Diamond}\alpha$ if and only if $w \Vdash^{\mathscr{M}} \alpha$ for some w such that $(w, w') \in R$.

The definition of entailment for sentences with the converse operators $\breve{\Box}$ and $\breve{\Diamond}$ follow that of Definition 2.5.

Now we are ready for the definition of modal pertinent entailment:

Definition 3.2 *α pertinently entails β in the* KT*-model $\mathscr{M}$* (noted $\alpha \mathrel{|\!\!<}^{\mathscr{M}} \beta$) if and only if $\alpha \models^{\mathscr{M}} \beta$ and $\beta \models^{\mathscr{M}} \breve{\Diamond}\alpha$. *$\alpha$ pertinently entails β in the class $\mathscr{C}$* of KT-models (noted $\alpha \mathrel{|\!\!<}^{\mathscr{C}} \beta$) if and only if for every $\mathscr{M} \in \mathscr{C}$, $\alpha \mathrel{|\!\!<}^{\mathscr{M}} \beta$.

Proposition 3.3 *Given a class of* KT*-models $\mathscr{C}$, $\mathrel{|\!\!<}^{\mathscr{C}} = \bigcap\{\mathrel{|\!\!<}^{\mathscr{M}} \mid \mathscr{M} \in \mathscr{C}\}$.*

Once again, when the class of KT-models we are working with is clear from the context, we shall dispense with superscripts and write $\alpha \mathrel{|\!\!<} \beta$ instead of $\alpha \mathrel{|\!\!<}^{\mathscr{C}} \beta$.

Intuitively, Definition 3.2 states that premiss α and consequence β are *mutually pertinent* if and only if α entails β and every β-world is accessible from *some* α-world — importantly, the $\beta \wedge \neg\alpha$-worlds (Figure 1). (The α-worlds are each accessible from itself.)

In the symbol $\mathrel{|\!\!<}$, the '$<$' refers to the infra-modal aspect of the entailment, as opposed to the '$=$' in $\models$, since what we do, in a sense, with the extra condition in Definition 3.2, is to 'cull down' some of the pairs in $\models$, obtaining a subset thereof.

One of the consequences of defining pertinence modally is that none of the

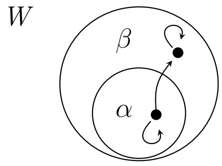

Fig. 1. Mutual pertinence of premiss α and consequence β: α-worlds are β-worlds and *any* β-world is accessible from *some* α-world.

connectives has a purely extensional semantics. Therefore our definition is not open to one of the criticisms levied against early relevance logics, namely that interpreting conjunction and disjunction connectives in a purely truth-functional way leads to a number of counter-intuitive results (cf. Remark 1.3).

We note that $\mid\!<$ can be defined equivalently, but more concisely and elegantly:

Proposition 3.4 *$\alpha \mid\!< \beta$ if and only if $\alpha \vee \beta \equiv \beta \wedge \breve{\Diamond}\alpha$.*

Given a premiss α, the set of consequences that α entails in our new relation are all the βs that lie between that particular α-premiss and $\breve{\Diamond}\alpha$, and hence form a sub-lattice (closed under conjunction and disjunction) of the Lindenbaum-Tarski algebra of the modal language [12, pp. 123–125], as depicted in Figure 2 below (with modal entailment going 'up').

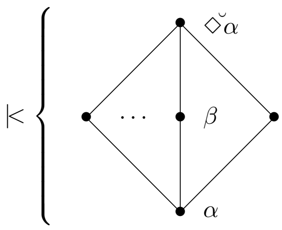

Fig. 2. The sub-lattice induced by pertinent entailment from premiss α.

Given a consequence β, the set of all those premisses α such that $\alpha \mid\!< \beta$ does not always constitute a sub-lattice of the Lindenbaum-Tarski algebra, since it is not, in general, closed under conjunction (cf. Section 6). But it is closed under disjunction: if $\alpha_1 \mid\!< \beta$ and $\alpha_2 \mid\!< \beta$, then $\alpha_1 \vee \alpha_2 \mid\!< \beta$.

The second part in Definition 3.2 adds 'pertinence' to the traditional modal entailment. It says: from every β-world we can look back to some world, possibly different from where we are, and from which we could have come, in which α is true. The pertinence resides in the fairly subtle relationship required between (i) the truth values of sentences, and (ii) the accessibility between worlds. Obviously, $\mid\!<$ is an infra-modal entailment relation: if $\alpha \mid\!< \beta$, then $\alpha \models \beta$.

Given a model $\mathscr{M} = \langle W, R, V \rangle$, $id_W \subseteq R \subseteq W \times W$. The *minimum* (with respect to $\subseteq$) case, i.e., in any subclass $\mathscr{C}$ of KT-models $\mathscr{M} = \langle W, R, V \rangle$ such that $R = id_W$, corresponds to the *maximum pertinence* of the relation $\mid\!<$, namely the case $\mid\!<\, = \,\equiv$ (i.e., logical equivalence), since now $\beta \models \breve{\Diamond}\alpha$ says that $\beta \models \alpha$. On the other hand, let $\models_<$ denote $\models \setminus \{(\bot, \beta) \mid \beta \not\equiv \bot\}$. Then the *maximum* case, i.e., in any subclass $\mathscr{C}$ of KT-models $\mathscr{M} = \langle W, R, V \rangle$ such that $R = W \times W$, corresponds to the *minimum pertinence* of $|\!<$, namely when $|\!< = \models_<$ (since now $\beta \models \breve{\Diamond}\alpha$ says that $\beta \not\equiv \bot$ implies $\alpha \not\equiv \bot$). Therefore we have:

Theorem 3.5 $\equiv\, \subseteq\, |\!<\, \subseteq\, \models_<$.

Notice that reflexivity of R is required in the proof of $\equiv\, \subseteq\, |\!<$ in Theorem 3.5. That is why we have chosen to work with KT-models.

4 Properties of Modal Pertinent Entailment

Now we discuss some of the properties of our pertinent entailment relation $|\!<$. We have already seen that $|\!<$ is the entailment $\models_<$ in the class of KT-models with a *total* accessibility relation, and $\equiv$ in the class of models with $R = id_W$. Therefore, $|\!<$ gives us a whole *spectrum* of entailment relations ranging between $\equiv$ and $\models_<$ (Theorem 3.5).

Non-explosiveness $|\!<$ is *non-explosive* in the strong sense that *falsum* is not omnigenerating, in fact, only self-generating: if $\bot\, |\!<\, \beta$, then $\beta \equiv \bot$. No contingent or tautological sentence is $|\!<$-entailed by a contradiction. This follows from the fact that for $\bot\, |\!<\, \beta$ to hold, $\beta \models \breve{\Diamond}\bot$ has to be the case, which holds only when $\beta \equiv \bot$. Note that this weak form of paraconsistency does not involve any metaphysical ideas (cf. Remark 1.4), and that all contradictions are equivalent.

More generally, we have the following:

Theorem 4.1 *Let* $\alpha\, |\!<^{\mathscr{C}}\, \beta$*. Then if* $\models^{\mathscr{C}} \alpha \rightarrow \bot$*, then* $\models^{\mathscr{C}} \beta \rightarrow \bot$*.*

In other words, no sentence satisfiable in a class $\mathscr{C}$ of models is $|\!<^{\mathscr{C}}$-entailed by a sentence *unsatisfiable* in that class.

Our pertinent entailment relation is *paratrivial* in the sense that *verum* is not omnigenerated, but only from premisses with very special properties. Consider $\alpha\, |\!<\, \top$ with an α which is not valid (in the underlying $\mathscr{C}$). From $\alpha\, |\!<\, \top$, we get $\top \models \breve{\Diamond}\alpha$, and then it follows that every world, in particular every $\neg\alpha$-world, is accessible from some α-world — indeed a rather strong stricture on α (and R). Intuitively, the assumption that from the α-worlds collectively *every* world whatsoever can be accessed justifies the mutual pertinence of α and $\top$.

Disjunctive Syllogism Classical *disjunctive syllogism* — $(\neg\alpha \vee \beta) \wedge \alpha \models \beta$, which is equivalent to $\beta \wedge \alpha \models \beta$ — is a minor pet hate of some relevance and relevant logicians: "the disjunctive syllogism is the only conceivable problematic rule of inference [amongst those under consideration]", [16, p. 33]. Even though classically we have no problem with $(\neg\alpha \vee \beta) \wedge \alpha \models \beta$, one can appreciate that in $\beta \wedge \alpha \models \beta$ the α is rather irrelevant. Does $|\!<$ help to isolate some 'relevant' (pertinent) cases of disjunctive syllogism?

$\beta \wedge \alpha\, |\!<\, \beta$ means that $\beta \wedge \alpha \models \beta$ and $\beta \models \breve{\Diamond}(\beta \wedge \alpha)$. We then have that to every β-world one can come from some (possibly other) $\beta \wedge \alpha$-world. Every β-world, even if not an α-world, can be reached from some $\beta \wedge \alpha$-world. This establishes the pertinence of β and α to each other. (This also means that if β is consistent, then so is $\beta \wedge \alpha$ — but that we already know from the strong non-explosiveness of $|\!<$.)

$(\neg\alpha \vee \beta) \wedge \alpha \models \beta$ is a version of *modus ponens*, viz. the *resolution rule* — while there are at least four different versions of *modus ponens* [11, p. 51]. For $|\!<$ we then saw that it holds only in a restricted and controlled way. A general form of *modus ponens* for $|\!<$ is proved further below.

Tautologies Another interesting property of $|\!<$ is that the set of *pertinent tautologies* of the modal language is identical to the set of all modal tautologies:

Theorem 4.2 $\top \,|\!<\, \alpha$ *if and only if* $\top \models \alpha$.

All valid formulas (in particular all tautologies), irrespective of their syntactical form, are semantically equivalent, and trivial, being satisfied in all of W (for each model in the class under consideration). This means that they do not exclude any possibility and contain no semantic information whatsoever. From a semantic point of view there is no justification for accepting some tautologies (say $\alpha \vee \neg\alpha$) but then rejecting others (say $\alpha \to (\beta \to \alpha)$). Syntactically different sentence forms without any difference of their model sets, of semantic meaning, are not treated differently and can be substituted anytime and anywhere by each other in a semantic approach. All the tautologies together are just one undifferentiated element $\top$ in the modal Lindenbaum-Tarski algebra of propositions [12], or, if you like, logical equivalence classes of sentences. Relevance/pertinence only makes sense relative to some extra semantic information (whether reflected on the object/syntactic level in a sentence or available only on the semantic/meta-level), while undifferentiated W has none. Pertinent entailment needs to move out of the domain of triviality, of tautologies, of "huh? — we know nothing!"

Contraposition Classically and modally we have *contraposition*: $\alpha \models \beta$ is equivalent to $\neg\beta \models \neg\alpha$. Not so for $|\!<$, and proof by contradiction does not hold in general. $\neg\beta \,|\!<\, \neg\alpha$ says that $\alpha \models \beta$ and $\neg\alpha \models \breve{\Diamond}\neg\beta$: Every α-world *is* a β-world and every $\neg\alpha$-world *can be reached from* some $\neg\beta$-world. This may be an entailment relation worthy of study, but which we shall not pursue further in this paper.

Deduction Theorem Now one question that naturally arises is whether the classical meta-theorem called *deduction*, or by some authors the *Ramsey test* for conditionals [11] ($\alpha \models \beta$ is equivalent to $\top \models \alpha \to \beta$), also holds for $|\!<$. So, is it the case that $\alpha \,|\!<\, \beta$ if and only if $\top \,|\!<\, \alpha \to \beta$?

For the left-to-right direction, suppose that $\alpha \,|\!<\, \beta$, i.e., $\alpha \models \beta$ and $\beta \models \breve{\Diamond}\alpha$. Then $\top \models \alpha \to \beta$ and surely $\alpha \to \beta \models \breve{\Diamond}\top$ (since the accessibility relation R has been assumed to be reflexive). Now, for the right-to-left direction, let us assume that $\top \,|\!<\, \alpha \to \beta$, i.e., $\top \models \alpha \to \beta$ and $\alpha \to \beta \models \breve{\Diamond}\top$. The second statement is just the triviality $\alpha \to \beta \models \top$. We do not (in general) get the needed $\beta \models \breve{\Diamond}\alpha$.

Hence, $\alpha \,|\!<\, \beta$ implies $\top \,|\!<\, \alpha \to \beta$, but not conversely — unless every β-world is accessible from some α-world, which is precisely the pertinence aspect of the definition of $|\!<$.

We noted in Theorem 4.2 above that the sets of modal and of pertinent tautologies are identical. While modal entailment $\alpha \models \beta$ is equivalent to $\alpha \to \beta$ being valid, this is false for pertinent entailment. For the latter, "to harp on the danger" of conflating entailment and conditional is indeed pertinently *not* "philosophically suspect" (remember Remark 1.1).

In our approach it is not difficult to define a modal conditional connective which *does* satisfy the Ramsey test. We define the modal binary connective $\diamond\!\!\rightarrow$, called the *pertinent conditional*, as follows:

Definition 4.3 $\alpha \diamond\!\!\rightarrow \beta \equiv_{\text{def}} (\alpha \rightarrow \beta) \wedge (\beta \rightarrow \breve{\Diamond}\alpha)$.

Theorem 4.4 *$\alpha \mid\!< \beta$ if and only if $\mid\!< \alpha \diamond\!\!\rightarrow \beta$.*

Positive Paradox One of the specific *bêtes noires* of relevance and relevant logicians is what they call *positive paradox* and write as $\alpha \rightarrow (\beta \rightarrow \alpha)$. With the introduction of our (stricter) conditional $\diamond\!\!\rightarrow$, one question that naturally arises is whether we have a pertinent version of positive paradox. The answer, as expected, is 'no', as shown by the following result:

Proposition 4.5 $\not\mid\!< \alpha \diamond\!\!\rightarrow (\beta \diamond\!\!\rightarrow \alpha)$.

Corollary 4.6 $\alpha \not\mid\!< \beta \diamond\!\!\rightarrow \alpha$.

With regards to a proof theory for $\mid\!<$, i.e., a sound and complete syntactical counterpart for our semantic entailment, we can resort to existing decision procedures, notably tableaux [19] and resolution [15], for both conditions in Definition 3.2. We do not develop this further here; however we do observe that pertinent entailment $\mid\!<$ satisfies the rule *modus ponens* (or disjunctive syllogism, cf. previous discussion) in the following sense:

Modus Ponens

$$\frac{\alpha \mid\!< \beta,\ \alpha \mid\!< \beta \rightarrow \gamma}{\alpha \mid\!< \gamma}$$

Moreover, it turns out that our modal pertinent entailment is *non-monotonic*:

Non-Monotonicity For the entailment $\mid\!<$, the following monotonicity rule *fails*:

$$\frac{\alpha \mid\!< \beta,\ \gamma \models \alpha}{\gamma \mid\!< \beta}$$

So, assuming $\alpha \mid\!< \beta$, we have *no* guarantee that $\alpha \wedge \alpha' \mid\!< \beta$: some β-world may not be accessible from *any* $\alpha \wedge \alpha'$-world, even though it is accessible from some α-world. (Remember that we have already discussed *disjunctive syllogism*, where we saw that $\beta \wedge \alpha \mid\!< \beta$ holds only in very special pertinent cases.)

This result stands in contrast to one of the fundamental, albeit tacit, assumptions in the non-monotonic reasoning literature [9,10,20,21], viz. that non-monotonic entailment relations are *a priori* supra-classical, or, at least, are obtained by relaxing the underlying (possibly non-classical) monotonic entailment [3].

Substitution of Equivalents Let $\alpha \mid\!< \beta$ and γ be a subformula of α. Then, for every γ' such that $\models \gamma \leftrightarrow \gamma'$, we have that $\alpha' \mid\!< \beta$, where α' is obtained by uniformly substituting γ' for γ in α. Consequently, we do not have the *variable sharing property* required in relevance logics [16].

We finish this section with a useful observation: By also requiring the underlying class of models to be *transitive*, i.e., if instead of KT we work in the modal logic S4 [13], then we get a pertinent entailment that satisfies some additional, and contextually desirable rules, notably:

Transitivity (Pertinent Left Strengthening)

$$\frac{\alpha \mathrel{|\!\!<} \beta,\ \beta \mathrel{|\!\!<} \gamma}{\alpha \mathrel{|\!\!<} \gamma}$$

Furthermore, if we work in S4, then the consequence operator $Cn : \mathscr{P}(\mathcal{L}) \longrightarrow \mathscr{P}(\mathcal{L})$ corresponding to $\mathrel{|\!\!<}$ and defined by $Cn(\Sigma) := \{\beta \mid \alpha \mathrel{|\!\!<} \beta \text{ for some } \alpha \in \Sigma\}$ is a *closure operator* [14].

5 Examples of Pertinent Reasoning

In this section we present and analyze a couple of examples where the notion of pertinence is involved. In the first example we give an account of the 'paraconsistent' character of $\mathrel{|\!\!<}$.

Example 5.1 Let p be interpreted as the statement "Mars orbits the Sun", and q as the statement "a red teapot is orbiting Mars", and let $\mathcal{B} = \{\neg p \rightarrow \Box\neg p\}$ be a set of background assumptions. (Intuitively we can think of $\mathcal{B}$ as saying that "if p were false, then you are stuck in $\neg p$-worlds".) We use $\mathcal{B}$ to define a class of KT-models in the following way: given a KT-model $\mathscr{M} = \langle W, R, V\rangle$, remove from $W \times W$ all links from $\neg p$-worlds to a p-world. Figure 3 shows an example of a model constructed in this way. (Since we consider local consequence, in this example, as well as in the next one, we use just a single model as illustration. Pertinent entailment in the subclass $\mathscr{C}$ of KT-models consistent with $\mathcal{B}$ then follows by generalizing over all models in $\mathscr{C}$ — cf. Definition 2.5.)

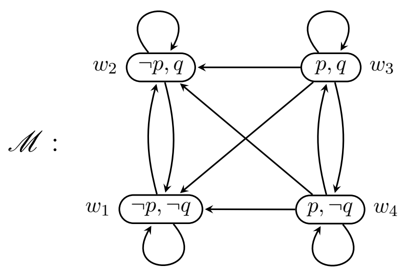

Fig. 3. A model induced by background assumptions $\mathcal{B} = \{\neg p \rightarrow \Box\neg p\}$.

One possible intuition behind the accessibility relation as defined in this example is that it restricts entailments from premisses that conflict with $\mathcal{B}$. In particular, no entailment from premiss α, with α conflicting with $\mathcal{B}$, to conclusion β, with β consistent with $\mathcal{B}$, is allowed.

When the premiss is compatible with background assumption $\mathcal{B}$, the entailment coincides with standard modal entailment on contingent formulas. For example, the following entailments are valid: $p \wedge q \mathrel{|\!\!<} p$; $p \wedge q \mathrel{|\!\!<} q$; $q \wedge \Diamond\neg p \mathrel{|\!\!<} q$; $\Diamond p \wedge q \mathrel{|\!\!<} q$; $p \mathrel{|\!\!<} p \vee q$; $q \mathrel{|\!\!<} p \vee q$; $\Diamond p \mathrel{|\!\!<} \Diamond p \vee q$; $p \mathrel{|\!\!<} \top$; and $\Box\Diamond q \mathrel{|\!\!<} \top$.

However, when the premiss is not compatible with the background assumptions, the entailment relation is restricted: if α contradicts $\mathcal{B}$, and $\alpha \mathrel{|\!\!<} \beta$, then β also contradicts $\mathcal{B}$. This illustrates the strong non-explosiveness of $\mathrel{|\!\!<}$ by the sterility of premisses contradicting background assumptions (cf. discussion in Section 4 and Theorem 4.1). For example, none of the following entailments hold: $\neg p \wedge \Diamond p \mathrel{|\!\!<} \neg p$; $q \wedge \Box p \mathrel{|\!\!<} q$; and $\Box p \mathrel{|\!\!<} \Box p \vee \neg q$.

Of course, entailments whose premisses contradict background assumptions are not the only ones ruled out by $\mathrel{|\!\!<}$. The following do not hold either: $\neg p \wedge q \mathrel{|\!\!<} q$; $\neg p \mathrel{|\!\!<} \neg p \vee q$; $p \wedge \Box\Box\neg p \mathrel{|\!\!<} p$; $q \wedge \Box\neg p \mathrel{|\!\!<} q$; and $\neg p \mathrel{|\!\!<} \top$. By forcing $\neg p$-worlds to remain stuck among themselves, we are implicitly declaring a 'preference' for p-worlds. What happens elsewhere is not pertinent.

The following example illustrates the links between pertinence and a notion of *causation*:

Example 5.2 Let us now consider the following variant of the Yale shooting problem in reasoning about actions, called the Walking Turkey Scenario [5]: Assume that we want to hunt a turkey, which may be alive or not, and which may either be walking around or not. In such a scenario we have one action, namely that of shooting the turkey with a gun. Our language has the propositions $\mathfrak{P} = \{s, a, w\}$. Let s be interpreted as "the turkey is shot"; a as "the turkey is alive"; and w as "the turkey is walking".

This time, our set of background assumptions is $\mathcal{B} = \{w \rightarrow a, s \rightarrow \neg a, \Diamond s\}$. The intuition behind $\mathcal{B}$ is that a walking turkey is alive; a shot turkey is dead; and it is possible to shoot the turkey.

Now suppose that we want to define a class of transitive models (cf. end of Section 4) in which the background assumptions in $\mathcal{B}$ are valid. First we make sure that the axiom schema 4 ($\Box\alpha \rightarrow \Box\Box\alpha$) [13] holds, and then we cull down the transitive models in which the formulas in $\mathcal{B}$ are *not* valid. One of the resulting models is depicted in Figure 4.

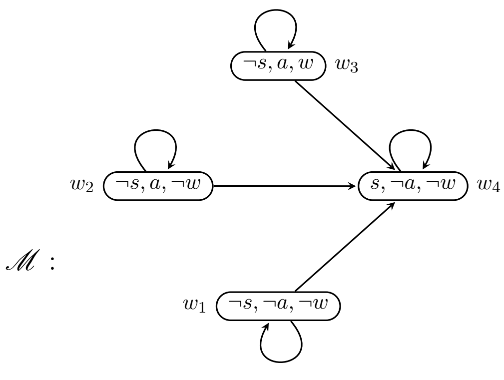

Fig. 4. A model induced by transitivity and background assumptions $\mathcal{B} = \{w \rightarrow a, s \rightarrow \neg a, \Diamond s\}$.

Here we are interested in entailments of the form: given that β is observed, is α the *cause* of β? While classically we have $\neg a \wedge \neg w \models \neg a$ and $\neg a \wedge \neg w \models \neg w$, we now get $\neg a \wedge \neg w \mathrel{|\!\!<} \neg a$, but $\neg a \wedge \neg w \not\mathrel{|\!\!<} \neg w$: the turkey could be alive and still prefer not to walk! (Remember Ockham's razor.)

Similar to our previous example, we do not have $a \wedge \Box\neg s \mathrel{|\!\!<} a$ (explanation incompatible with background assumption — cf. Theorem 4.1). On the other hand, we do have $a \wedge \Box\Diamond s \mathrel{|\!\!<} a$ (substitution of equivalents, since $\Box\Diamond s$ is valid in this class of models). Moreover, we have neither $s \mathrel{|\!\!<} \neg a$ nor $\neg a \mathrel{|\!\!<} s$: being shot is not the only possible cause for the death of the turkey; and being already dead does not explain anything at all for being shot.

6 Discussion and Related Work

Most of the existing work on substructural logics have mainly focused on weakening the generating engine of *axioms* and *inference rules* to get rid of unwanted entailments [16,25]. Other existing approaches are algebraic in nature [18]. For those reasons, the referred works are not directly comparable to ours.

Quite recently, after an early start [26], a few publications concentrated on further developing a proper semantics for relevant logics [23]. There, a possible worlds approach is also defined, but with the aid of a *ternary* accessibility relation between worlds. Having (w, w', w'') in the accessibility relation means different things for different authors. For Meyer "[w]orlds are best demythologized as theories", and then, paraphrasing, theory w'' consists of all the outputs got by applying *modus ponens* in a certain way to major premisses from w and minor premisses from w'. Here we propose a simpler approach, viz. via a *binary* accessibility relation on plain W, for carrying out the required construction of pertinence.

Meyer claimed that in almost all standard relevance logics the *relevant* conditional (usually written as $\rightarrow$) *cannot* be defined as a modalised truth function [22]. More explicitly, Meyer proved that no standard *relevant* conditional can be represented as a "*strict*" $\Box\phi(\alpha, \beta)$, where $\phi(\alpha, \beta)$ is any truth functional combination of α and β. This prescription that "modalizing" in this context must mean "having $\Box$ as main connective" is of course restrictive. Our modal treatment in Definition 4.3 and Theorem 4.4 of the *pertinent* conditional connective $\diamond\!\!\rightarrow$ shows that by lifting pertinence to the meta-level we can achieve the desired result in a still quite elegant way, even if not with the main operator $\Box$.

Research on substructural logic usually adopts a *bottom-up* strategy (going from 'nothing' up to the entailments considered as relevant), and quite often via a proof-theoretic approach. Here we have followed a semantic-based *top-down* strategy: we start from full modal logic KT and then go down to infra-modal consequence by culling the impertinent entailments. A similar strategy is that of 'filtering out' undesirable classical entailment pairs to prevent 'explosion' (*ex contradictione quodlibet*) in some paraconsistent logics [24, pp. 297–299].

More in-depth research remains to be done in relating our work to existing notions of non-monotonic inference. However for present purposes we suffice with an observation contrasting our pertinent conditional (Definition 4.3) with Cantwell's defeasible conditional [11]:

Cantwell also argues (with many examples) for an indispensable distinction between his $\alpha \mathrel{|\!\sim} \beta$ (from the *assumption* or supposition α the consequence β has to be accepted) and his $\mathrel{|\!\sim} \alpha \rightarrow \beta$ (irrespective of any assumption, the conditional

$\alpha \to \beta$ *is accepted*). But the context and nature of his $\mid\!\sim$ and $\to$ — with the purpose of discriminating strictly between *supposing* and *accepting* a statement — lead to properties very different from those of our $\mid\!<$: in most interesting cases Cantwell's valid $\alpha \mid\!\sim \beta$ does *not* imply a valid $\mid\!\sim \alpha \to \beta$. Both his $\mid\!\sim$ and his $\to$ are non-classical, while our $\to$ is strictly classical (cf. Definition 4.3 for a non-classical version thereof); and his $\mid\!\sim$ is supra-classical.

7 Concluding Remarks

In this paper we have shown that a semantic notion of pertinence can be captured elegantly using a simple modal logic. The information about pertinence is *not* directly in the accessibility relation between worlds (nor in the modal operator representing it). Instead, it is in the meta statement "α and β are mutually pertinent", which here we have formalized as the extra condition $\beta \models \breve{\Diamond}\alpha$.

We have seen (Theorem 3.5) that a modal approach allows for a whole spectrum of pertinent entailments, ranging between $\equiv$ and $\models_<$, and offering potential reasoning tools for many different contexts (cf. Remark 1.5 and examples in Section 5). We are currently investigating the possibility of capturing with our framework notions such as *obligation* and *belief*.

Our pertinent entailment relation $\mid\!<$ restricts some paradoxes shunned by relevance and relevant logics in an interesting way. Moreover, we have shown that $\mid\!<$ also possesses other non-classical properties, like strong non-explosiveness and non-monotonicity. We have also seen that $\mid\!<$ satisfies its corresponding version of *modus ponens*. For a discussion on the other inference rules traditionally considered in the literature that are also satisfied by pertinent entailment, see the work by Britz *et al.* [10].

In Examples 5.1 and 5.2 we illustrated how entailment may be curtailed by background assumptions by restricting the class of accessibility relations. Thus background assumptions may prevent unwanted entailments of a formula from a premiss. In this sense, our logic shares some aspects of *adaptive logics*, a family of non-monotonic logics characterized by a dynamic proof theory [6].

In this paper we have investigated the case of infra-modal consequence, which is in the spirit of traditional substructural logics like relevance and relevant logics. We plan to pursue future work by investigating further cases as well as the *supra-modal* counterparts of our entailment relations, which relate to prototypical reasoning and other forms of venturous reasoning.

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