A Connection Method for a Defeasible Extension of $\mathcal{ALC}$

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Abstract. In this paper, we propose a connection method à la Bibel for an exception-tolerant extension of $\mathcal{ALC}$. As for the language, we assume $\mathcal{ALC}$ extended with a typicality operator on concepts, which is a variant of defeasible DLs studied in the literature over the past decade and in which most of these can be embedded. We revisit the definition of matrix representation of a knowledge base and establish the conditions for a given axiom to be provable from it. In particular, we show how term substitution is dealt with and define a suitable condition of blocking in the presence of typicality operators. We show that the calculus terminates and that it is sound and complete w.r.t. a DL version of the preferential semantics widely adopted in non-monotonic reasoning.

Keywords: Description logic · defeasible reasoning · connection method

1 Introduction

The problem of modelling exceptions in ontologies and reasoning meaningfully in their presence has received a great deal of attention over the past decade. Among the emblematic approaches put forward in the literature feature Giordano et al.'s description logics of typicality [24,25,28], Britz et al.'s defeasible subsumption relations [9,10,8], Bonatti et al.'s light-weight DLs of normality [5,4,6], besides Casini and Straccia's seminal work on the computational counterpart of non-monotonic entailment in DLs [18,19] and its implementation [17]. These investigations have given rise to a whole family of defeasible description logics of varying expressive power and with the ability to handle exceptions at both the modelling and reasoning levels in a number of ways [3,11,12,13,16,20,30].

One of the interesting features of some of the aforementioned approaches is the fact that, depending on the underlying DL that is assumed and given certain conditions on how exceptionality (or typicality) is expressed, the kind of non-monotonic reasoning that is performed can be reduced to (a polynomial number of calls to) classical entailment check. Therefore, the study of automated deduction for the various flavours of defeasible DLs, as well as its potential reduction to classical reasoning, remains a relevant research topic in the area.

The development of proof methods for defeasible description logics has, in a sense, followed on the footsteps of those for classical DLs. As a result, the majority of existing decision procedures for reasoning with defeasible ontologies are based on semantic tableaux [12,14,25,27,31]. Notwithstanding the commonly extolled virtues of tableau systems, there are equally viable alternatives in the literature on automated theorem proving (ATP). One prominent example is the connection method (CM), defined by Bibel in the late '70s [2], which earned good reputation in the field of ATP in the '80s and '90s and has enjoyed a recent revival in the context of (classical) modal and description logics [21,22].

The connection method consists in a direct proof procedure of which the main internal structure is a matrix representation of the knowledge base and associated query. It was designed having as a guideline a more parsimonious usage of memory during proof search. Indeed, contrary to tableaux and resolution, the connection method does not create intermediate clauses or sentences along the way, keeping its search space confined to the boundaries of the matrix it started off with. The first connection calculus for description logics, $\mathcal{ALC}$ θ-CM, has been proposed by Freitas and Otten [21]. It incorporates several features of most DL proof systems such as blocking, absence of variables, unification and Skolem functions. Moreover, a C++ implementation of the calculus, Raccoon [29], has been developed.[5] Worthy of mention is the fact that, in spite of not incorporating any of the optimisations commonly done for DL tableaux systems, Raccoon performed competitively when reasoning over $\mathcal{ALC}$ ontologies in comparison to cutting-edge highly-optimised tableau-based reasoners which had ranked high in past competitions of DL reasoners (see `https://goo.gl/V9Ewkv` for details).

The present paper aims to make the first steps in the design of connection methods for reasoning over defeasible ontologies. In particular, we aim at providing a concrete calculus for a version of defeasible DL frequently considered in the literature that can be used to endow Raccoon with basic non-monotonic reasoning features. We hope our constructions will serve as a springboard for the development of the connection method in defeasible DLs of varying expressive power and further extensions of Raccoon.

In this paper, we shall assume the reader's familiarity with the different DL families, in particular with $\mathcal{ALC}$. The remaining of the text is structured as follows: in Section 2, we present the defeasible extension of $\mathcal{ALC}$ we build on in this work; Section 3 is the heart of the paper and introduces our connection method for reasoning with defeasible ontologies, of which the inner workings are illustrated with a worked-out example in Section 4; Section 5 concludes the paper with a discussion and possible directions for further investigation.

[5] `https://github.com/dmfilho/raccoon`

2 Preliminaries: The Defeasible Description Logic $\mathcal{ALC}^{\bullet}$

The defeasible DL $\mathcal{ALCH}^{\bullet}$ [31] is an extension of $\mathcal{ALCH}$ with typicality operators on both complex concepts and role names. Intuitively, a concept expression of the form $\bullet C$ denotes the most typical (alias normal) objects in the class C, whereas a role expression of the form $\bullet r$, with r an atomic role, denotes the most typical instances of the relationship represented by r. To give a glimpse of $\mathcal{ALCH}^{\bullet}$'s expressive power, the TBox axioms $\bullet\mathsf{Muggle} \sqsubseteq \neg\mathsf{Wizard}$, $\mathsf{HalfBloodWizard} \sqsubseteq \exists\mathsf{casts.Spell}\sqcap\neg\bullet\mathsf{Wizard}$, and $\bullet\mathsf{Wizard} \sqsubseteq \exists\bullet\mathsf{attachedWith.Wand}$ specify, respectively, that "typical muggles are not wizards", "half-blood wizards cast spells but are not typical wizards", and "typical wizards have a typical attachment with a wand". The RBox axiom $\mathsf{masterOf} \sqsubseteq \bullet\mathsf{attachedWith}$ states that "to be a master of (a wand) means to be typically attached with (that wand)". Furthermore, the ABox assertion $\neg\bullet\mathsf{attachedWith}(\mathsf{lordVoldemort}, \mathsf{elderWand})$ formalises the intuition that $\mathsf{lordVoldemort}$ is not typically attached with $\mathsf{elderWand}$.

In this paper, we shall nevertheless focus on the fragment $\mathcal{ALC}^{\bullet}$ of $\mathcal{ALCH}^{\bullet}$ without role hierarchies and allowing for typicality of concepts, only. The reason for this choice is three-fold: first, it simplifies many of the technical definitions and results to be presented below; second, the fragment we here consider is strongly related to the defeasible DL $\mathcal{ALC}+\mathbf{T}$ of Giordano et al. [24,25,28] and to the extension of $\mathcal{ALC}$ with defeasible subsumption by Britz et al. [8,9,10], which have become, in a sense, the prototypical defeasible DLs in the recent literature, and third, the definition of a proof method for this fragment not based on semantic tableaux is a problem that is interesting in its own right. (Needless to say, the work we report on here can naturally serve as a springboard for the definition of connection methods for more expressive defeasible DLs.)

The (finite) sets of concept, role and individual names are denoted, respectively, with C, R, and I. With $A, B, \ldots$ we denote atomic concepts, with $r, s, \ldots$ role names, and with $a, b, \ldots$ individual names. Complex concepts of $\mathcal{ALC}^{\bullet}$ are denoted with $C, D, \ldots$ and are built according to the following grammar:

$$C ::= \mathsf{C} \mid \top \mid \bot \mid \neg C \mid C \sqcap C \mid C \sqcup C \mid \forall \mathsf{R}.C \mid \exists \mathsf{R}.C \mid \bullet C \quad (1)$$

The definitions of axiom, GCI, assertion, TBox and ABox are as in the classical $\mathcal{ALC}$ case. If $\mathcal{T}$ and $\mathcal{A}$ are, respectively, a TBox and an ABox, with $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ we denote henceforth a *knowledge base* (alias ontology).

Example 1 (Wizarding World Scenario). Assume we are interested in modeling facts about the wizarding world and its wonderful features. We have the atomic concepts $\mathsf{C} = \{\mathsf{Muggle}, \mathsf{Wizard}, \mathsf{Spell}\}$ representing, respectively, the class of muggles, wizards, and spells. As for the set of atomic roles, we have the singleton $\mathsf{R} = \{\mathsf{casts}\}$, representing a cast of a spell (done by a wizard). The set of individuals I is $\{\mathsf{hermione}, \mathsf{oculusReparo}\}$. Below we have an example of an $\mathcal{ALC}^{\bullet}$ knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ for the wizarding world scenario.

$$\mathcal{T} = \left\{ \begin{array}{c} \bullet\mathsf{Muggle} \sqsubseteq \neg\mathsf{Wizard}, \\ \exists\mathsf{casts.Spell} \sqsubseteq \mathsf{Wizard} \end{array} \right\} \qquad \mathcal{A} = \left\{ \begin{array}{c} \mathsf{Muggle}(\mathsf{hermione}), \\ \mathsf{Spell}(\mathsf{oculusReparo}), \\ \mathsf{casts}(\mathsf{hermione}, \mathsf{oculusReparo}) \end{array} \right\}$$

The semantics of $\mathcal{ALC}^{\bullet}$ extends that of classical $\mathcal{ALC}$ and is in terms of partially-ordered structures called ordered interpretations. Before introducing these, we recall a few notions.

A binary relation is a *strict partial order* if it is irreflexive and transitive. If $<$ is a strict partial order on a given set X, with $\min_{<} X \stackrel{\text{def}}{=} \{x \in X \mid \text{there is no } y \in X \text{ s.t. } y < x\}$ we denote the *minimal elements* of X w.r.t. $<$. A strict partial order on a set X is *well-founded* if for every $\emptyset \neq X' \subseteq X$, we have $\min_{<} X' \neq \emptyset$.

Definition 1 (Ordered interpretation). *An $\mathcal{ALC}^{\bullet}$* ***ordered interpretation*** *is a tuple $\mathcal{O} \stackrel{\text{def}}{=} \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, <^{\mathcal{O}} \rangle$ s.t. $\langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}} \rangle$ is a classical $\mathcal{ALC}$ interpretation, $<^{\mathcal{O}} \subseteq \Delta^{\mathcal{O}} \times \Delta^{\mathcal{O}}$ is a well-founded strict partial order.*

Given $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, <^{\mathcal{O}} \rangle$, the intuition of $\Delta^{\mathcal{O}}$ and $\cdot^{\mathcal{O}}$ is the same as in a standard $\mathcal{ALC}$ interpretation. The intuition underlying the ordering $<^{\mathcal{O}}$ is that it plays the role of a *preference relation* (or *normality* ordering): the objects that are lower down in the ordering $<^{\mathcal{O}}$ are deemed more normal (or typical) than those higher up in $<^{\mathcal{O}}$. Within the context of (the interpretation of) a concept C, $<^{\mathcal{O}}$ therefore allows us to single out the most normal representatives falling under C, which is the intuition of the semantics of concepts of the form $\bullet C$:

Definition 2 (Semantics). *An ordered interpretation $\mathcal{O} = \langle \Delta^{\mathcal{O}}, \cdot^{\mathcal{O}}, <^{\mathcal{O}} \rangle$ interprets the classical constructors in the usual way. Typicality-based concepts are interpreted as $(\bullet C)^{\mathcal{O}} \stackrel{\text{def}}{=} \min_{<^{\mathcal{O}}} C^{\mathcal{O}}$.*

Hence, to be a typical element of a concept amounts to being one of the most preferred elements in the interpretation of that concept.

The definition of *satisfaction* of a statement α by an ordered interpretation $\mathcal{O}$, denoted as $\mathcal{O} \Vdash \alpha$, carries over from the classical case. If X is a set of statements, with $\mathcal{O} \Vdash X$ we denote the fact $\mathcal{O}$ satisfies each statement in X, in which case we say $\mathcal{O}$ is a *model* of X. We say $\mathcal{O}$ is a model of a knowledge base $\mathcal{K} = (\mathcal{T}, \mathcal{A})$, denoted $\mathcal{O} \Vdash \mathcal{K}$, if $\mathcal{O}$ is a model of both $\mathcal{T}$ and $\mathcal{A}$.

Given a knowledge base $\mathcal{K}$ and a statement α, we say that $\mathcal{K}$ *preferentially entails* α, denoted $\mathcal{K} \models \alpha$, if, for every $\mathcal{O}$ such that $\mathcal{O} \Vdash \mathcal{K}$, we have $\mathcal{O} \Vdash \alpha$.

3 The Connection Method for $\mathcal{ALC}^{\bullet}$

The connection method (CM) [2] consists in a validity procedure (in opposition to refutation procedures such as tableaux and resolution), i.e., it tries to prove whether a formula (query) is valid directly. If $X = \{\varphi_1, \ldots, \varphi_n\}$ is a (finite) set of first-order formulae, then in order to check whether $X \models \psi$, for some first-order formula ψ, the validity of the formula $(\varphi_1 \wedge \ldots \wedge \varphi_n) \rightarrow \psi$ $(X \rightarrow \psi)$, i.e., of $\neg X \vee \psi$, must be proven. Hence, CM requires the conversion of formulae into the disjunctive normal form (DNF). Of course, when moving to the DL case, the crux of the matter is precisely how to express the negation of the knowledge base, along with doing away with variables and Skolem functions.

$\mathcal{ALC}$ θ-CM [21] was the first CM proposed for DLs. Among its main features is the fact it requires neither variables in the initial representation of the axioms nor Skolem functions. Moreover, it includes a blocking solution to ensure termination, as commonly done in the field, and the definition of θ-substitution as a suitable replacement for variable unification.

In this section, we introduce a connection method for $\mathcal{ALC}^\bullet$, which we henceforth call $\mathcal{ALC}^\bullet$ θ-CM. It inherits the $\mathcal{ALC}$ θ-CM blocking, and deals with typicality by building an auxiliary structure which stands for the preference relation.

3.1 Matrix Representation

The connection method requires formulae to be represented as matrices, composed of clauses, which are conjunctions of literals, possibly negated. To better handle concepts built using the typicality operator $\bullet$, we introduce below a typicality normal form (TNF).

Definition 3 (Pure disjunction and pure conjunction). *A* ***pure disjunction*** *($\check{D}$) and a* ***pure conjunction*** *($\hat{E}$) are recursively defined as follows:*

$$\check{D} ::= \mathsf{C} \mid \neg\mathsf{C} \mid \check{D} \sqcup \check{D} \mid \forall \mathsf{R}.\mathsf{C} \qquad \hat{E} ::= \mathsf{C} \mid \neg\mathsf{C} \mid \hat{E} \sqcap \hat{E} \mid \exists \mathsf{R}.\mathsf{C}$$

Pure disjunctions/conjunctions save memory and simplify proofs, see [21].

Definition 4 (Typicality normal form). *Let $\hat{E}$ be a pure conjunction, $\check{D}$ be a pure disjunction, r be a role name, A and B be atomic concepts, and a and b be individual names. A TBox axiom is in* ***typicality normal form*** *(TNF) if it is in one of the forms below:*

1. $\hat{E} \sqsubseteq \check{D}$;
2. $A \sqsubseteq \exists r.B$;
3. $\forall r.A \sqsubseteq B$;
4. $A \sqsubseteq \bullet B$;
5. $A \sqsubseteq \neg\bullet B$;
6. $\neg\bullet A \sqsubseteq B$;
7. $\bullet A \sqsubseteq B$.

An ABox axiom is in TNF if it is in one of the forms: (i) $A(a)$, (ii) $\neg A(a)$, (iii) $r(a, b)$, or $\neg r(a, b)$. A knowledge base (KB) is in TNF if all its axioms are in TNF. A query is in TNF if the queried axiom and the KB are in TNF.

The TNF allows us to deal only with axioms in specific forms so that our matrix representation is simple and straightforward. An example of this simplicity is the restriction on concept assertions: only assertions with atomic concepts are allowed. Whether typical concept assertions or other assertions with complex concepts are present in the ABox, these assertions can be rewritten and these concepts relocated to the TBox as a subsumption axiom.

Proposition 1. *Let $\mathcal{K}$ be an $\mathcal{ALC}^\bullet$ knowledge base. There exists an $\mathcal{ALC}^\bullet$ knowledge base $\mathcal{K}'$ in TNF such that if $\mathcal{O}$ is a model of $\mathcal{K}'$, then $\mathcal{O} \Vdash \alpha$ for all $\alpha \in \mathcal{K}$.*

Since the method is designed to check for the validity of a knowledge base (or query), we represent it as a 'disjunction set'. This set contains the original axioms negated, although DLs do not allow for the negation of axioms to be expressed. One way to solve this problem is to represent the knowledge base as a single set, instantiating the TBox axioms, so that the TBox can be dropped. This process is similar to the unfolding technique done by Baader et al. [1]; here we chose to keep variables in order to avoid unnecessarily long matrices.

Definition 5 (Negation of a knowledge base). *If $\mathcal{K} = (\mathcal{T}, \mathcal{A})$ is a knowledge base, its **negation** is the set*

$$\neg\mathcal{K} \overset{\text{def}}{=} \{\exists x(D \sqcap \neg E)(x) \mid D \sqsubseteq E \in \mathcal{T} \textit{ and } x \textit{ is a new variable}\} \cup \{\neg\gamma \mid \gamma \in \mathcal{A}\} \quad (2)$$

Definition 6 (Literal and clause). *A **literal** has one of the forms $A(x)$, $r(x, y)$, or their negations, where A is an atomic concept, r is an atomic role, and x, y are variables or individuals. A **clause** is a set of literals. A **<-clause** is a pair $C^< \overset{\text{def}}{=} \langle C, < \rangle$, where C is a clause and $<$ is an auxiliary relation on variables or individuals.*

The <-clauses are obtained from axioms containing a typicality operator, as per the rules in Table 1. A clause C which is not related to a relation $<$ can be seen as the <-clause $\langle C, \emptyset \rangle$.

Definition 7 (Matrix representation and graphical matrix). *Let $\mathcal{K}$ be a knowledge base and $\neg\mathcal{K}$ its negation. A **<-matrix** $M^<$ is the set of <-clauses mapped from the axioms in $\neg\mathcal{K}$ as shown in Table 1. The **graphical matrix** of $M^<$ displays clauses as columns; the relation $<$ (the union of the relations of all <-clauses in $M^<$) associated to the matrix is shown as a lattice next to it.*

Definition 8 (Path, connection, θ-substitution and θ-complementary connection). *A **path** through a <-matrix $M^<$ is a set composed by one literal of each clause in $M^<$. A **connection** is a pair of literals $\{E, \neg E\}$ with the same concept or role name, but complementary to each other. A θ-**substitution** assigns each variable with an individual or another variable. The application of a θ-substitution to a literal is an application to its variables, i.e., $\theta(E(x)) = E(\theta(x))$ and $\theta(r(x, y)) = r(\theta(x), \theta(y))$. A θ-**complementary** connection is a pair $\{E(x), \neg E(y)\}$ or $\{r(x, v), \neg r(y, u)\}$, where $\theta(x) = \theta(y)$ and $\theta(v) = \theta(u)$. The complement $\overline{L}$ of a literal L is $\neg E$, if $L = E$, and E, if $L = \neg E$.*

Definition 9 (Set of concepts). *If x is a variable or an individual, $\tau(x) \overset{\text{def}}{=} \{E \in \mathsf{C} \mid E(x) \in Path\}$ is the **set of concepts** of x containing all concepts related to x in a path.*

Definition 10 (Admissible θ-substitution, preference condition). *Let θ be a term substitution. We say θ is **admissible** if the preference condition holds. The condition is defined as follows: let x and y be variables or individuals in $M^<$, and $<$ be the union of the relations of all <-clauses in $M^<$. Then, the **preference condition** holds after a θ-substitution if $\{(x, y), (y, x)\} \cap <^+ = \emptyset$, with $<^+$ the transitive closure of $<$.*

Table 1. Normal forms and their matrix representation. Let $\hat{C}$ be a pure conjunction, $\check{D}$ be a pure disjunction, A and B be atomic concepts, r be a role name, x, x', y be variables, a be an individual name, and let e be a new individual name.

α	$<$-clausal form of $\neg\alpha$	Matrix Representation
$\hat{C} \sqsubseteq \check{D}$	$\langle\{C_1(x), \cdots, C_n(x),$ $\neg D_1(x), \cdots, \neg D_m(x)\}, \emptyset\rangle$	$\begin{bmatrix} C_1(x) \\ \vdots \\ C_n(x) \\ \neg D_1(x) \\ \vdots \\ \neg D_m(x) \end{bmatrix}$
$A \sqsubseteq \exists r.B$	$\langle\{A(x), \neg r(x,a)\}, \emptyset\rangle,$ $\langle\{A(x'), \neg B(a)\}, \emptyset\rangle$	$\begin{bmatrix} A(x) & A(x') \\ \neg r(x,a) & \neg B(a) \end{bmatrix}$
$\forall r.A \sqsubseteq B$	$\langle\{\neg r(x,a), \neg B(x)\}, \emptyset\rangle,$ $\langle\{A(a), \neg B(x')\}, \emptyset\rangle$	$\begin{bmatrix} \neg r(x,a) & A(a) \\ \neg B(x) & \neg B(x') \end{bmatrix}$
$A \sqsubseteq \bullet B$	$\langle\{A(x), \neg B(x)\}, \{(y,x)\}\rangle,$ $\langle\{A(x'), B(y)\}, \{(y,x), (y,x')\}\rangle$	$\begin{bmatrix} A(x) & A(x') \\ \neg B(x) & B(y) \end{bmatrix}$ $x \dashrightarrow y \dashleftarrow x'$
$\neg\bullet A \sqsubseteq B$	$\langle\{\neg A(x), \neg B(x)\}, \{(y,x)\}\rangle,$ $\langle\{A(y), \neg B(x')\}, \{(y,x), (y,x')\}\rangle$	$\begin{bmatrix} \neg A(x) & A(y) \\ \neg B(x) & \neg B(x') \end{bmatrix}$ $x \dashrightarrow y \dashleftarrow x'$
$A \sqsubseteq \neg\bullet B$	$\langle\{A(x), B(x), \neg B(e)\}, \{(e,x), (y,e)\}\rangle,$ $\langle\{A(x'), B(x'), B(y)\}, \{(e,x), (e,x'), (y,e)\}\rangle$	$\begin{bmatrix} A(x) & A(x') \\ B(x) & B(x') \\ \neg B(e) & B(y) \end{bmatrix}$ $x \dashrightarrow e \dashleftarrow x'$, $e \dashrightarrow y$
$\bullet A \sqsubseteq B$	$\langle\{A(x), \neg A(e), \neg B(x)\}, \{(e,x), (y,e)\}\rangle,$ $\langle\{A(x'), A(y), \neg B(x')\}, \{(e,x), (e,x'), (y,e)\}\rangle$	$\begin{bmatrix} A(x) & A(x') \\ \neg A(e) & A(y) \\ \neg B(x) & \neg B(x') \end{bmatrix}$ $x \dashrightarrow e \dashleftarrow x'$, $e \dashrightarrow y$
$A(a)$	$\langle\{\neg A(a)\}, \emptyset\rangle$	$\begin{bmatrix} \neg A(a) \end{bmatrix}$
$\neg A(a)$	$\langle\{A(a)\}, \emptyset\rangle$	$\begin{bmatrix} A(a) \end{bmatrix}$
$r(a,b)$	$\langle\{\neg r(a,b)\}, \emptyset\rangle$	$\begin{bmatrix} \neg r(a,b) \end{bmatrix}$
$\neg r(a,b)$	$\langle\{r(a,b)\}, \emptyset\rangle$	$\begin{bmatrix} r(a,b) \end{bmatrix}$

3.2 The Calculus

We now formally define the $\mathcal{ALC}^{\bullet}$ connection calculus ($\mathcal{ALC}^{\bullet}$ θ-CM). The novelty here with regard to the previous classical DL-oriented calculus $\mathcal{ALC}$ θ-CM is twofold: (i) the introduction of a new structure, the purpose of which is to denote a preference relation, and (ii) the preference condition, which checks whether, in a candidate connection, the variables involved (if any) are incomparable w.r.t. the auxiliary ordering; in this case, the connection is allowed. By doing so, the calculus conforms to the preferential semantics defined in Section 2.

Definition 11 (Multiplicity). *Let $M^{<}$ be a $<$-matrix and $C^{<}$ be a $<$-clause in $M^{<}$. The* ***multiplicity*** *is a function $\mu : M^{<} \longrightarrow \mathbb{N}$ that assigns to each $<$-clause in $M^{<}$ a natural number denoting the number of copies of that clause in the matrix. Let ${}^{i}C^{<}$ be the i-th copy of the clause $C^{<}$, $1 \leq i \leq \mu(C^{<})$. We say that $C^{<}_{\mu} \stackrel{\text{def}}{=} \{{}^{1}C^{<}, \cdots, {}^{\mu(C^{<})}C^{<}\}$ is the set of copies of $C^{<}$ and*

$$M^{<}_{\mu} \stackrel{\text{def}}{=} M^{<} \cup \bigcup_{C^{<} \in M^{<}} C^{<}_{\mu}$$

is the $<$-matrix $M^{<}$ combined with the clause copies.

Definition 12 (Blocking condition). *Let $\theta(x_{\mu})$ be a new individual, i.e., $\theta(x_{\mu}) \notin \mathsf{I}$. If $\tau(\theta(x_{\mu})) \subseteq \tau(\theta(x_{\mu-1}))$, then $\theta(x_{\mu})$ is* ***blocked****.*

Definition 13 ($\mathcal{ALC}^{\bullet}$ θ-CM calculus). *Let $M^{<}$ be a $<$-matrix and let $\alpha \in M$ be a $<$-clause (or the query $<$-clause). The rules of the $\mathcal{ALC}^{\bullet}$ connection calculus ($\mathcal{ALC}^{\bullet}$ θ-CM) are as in Figure 1.*

The rules in Figure 1 are applied in an analytical, bottom-up way and the basic structure is the tuple $\langle C, M^{<}, Path\rangle$, where clause C is the open subgoal, $M^{<}$ is the $<$-matrix, and $Path$ is the active path. When the Copy rule is applied, it has to be followed by the Extension or the Reduction rule. (Freitas and Varzinczak [22] give examples of copying and blocking and we shall not do so here.)

Definition 14 (Matrix validity). *Let $\mathcal{K}$ be a knowledge base and $M^{<}$ the $<$-matrix of the negation of $\mathcal{K}$. We say that $M^{<}$ is* ***valid*** *if $\mathcal{K} \models \bot$.*

Theorem 1 (Matrix characterization). *A $<$-matrix $M^{<}$ is valid iff there exists a multiplicity μ, an admissible θ-substitution and a set of connections S such that every path through $M^{<}_{\mu}$ contains a θ-complementary connection $\{\theta(L_1), \theta(L_2)\} \in S$.*

The tuple $\langle \mu, \theta, S\rangle$, where μ, θ, and S are as referred to in Theorem 1, is called an *ordered matrix proof.*

Definition 15 (Preferential connection proof). *Let $\langle C, M^{<}, Path\rangle$ be a connection calculus tuple. A* ***preferential connection proof*** *for this tuple*

$$\textit{Axiom (Ax)}\ \frac{}{\{\}, M^<, \textit{Path}}$$

$$\textit{Start Rule (St)}\ \frac{C_1, M^<, \{\}}{\epsilon, M^<, \epsilon},\ \text{with } C_1 \in \{C \mid \langle C, <' \rangle \in \alpha\}$$

$$\textit{Reduction Rule (Red)}\ \frac{C, M^<, \textit{Path} \cup \{L_2\}}{C \cup \{L_1\}, M^<, \textit{Path} \cup \{L_2\}},\ \text{with } \theta(L_1) = \theta(\overline{L_2})$$

$$\textit{Extension Rule (Ext)}\ \frac{C_1 \setminus \{L_2\}, M^<, \textit{Path} \cup \{L_1\} \quad C, M^<, \textit{Path}}{C \cup \{L_1\}, M^<, \textit{Path}},\ \text{with } \theta(L_1) = \theta(\overline{L_2})$$

$$\textit{Copy Rule (Cop)}\ \frac{C' \cup \{L_1\}, M^< \cup \{C_2^{<''}\}, \textit{Path}}{C \cup \{L_1\}, M^<, \textit{Path}},$$

where $C_2^{<''}$ is a copy of $C_1^{<'}, L_2 \in C_2^{<''}, \theta(L_1) = \theta(\overline{L_2})$, and the blocking condition holds

Fig. 1. The $\mathcal{ALC}^\bullet$ θ-CM calculus.

(also said the preferential connection proof for the $<$-matrix $M^<$) is the application of the rules of the calculus to the initial tuple $\langle \epsilon, M^<, \epsilon \rangle$ (i.e., with C and Path initially empty), using an admissible θ-substitution s.t. all leaves are axioms (Ax).

Theorem 2 (Termination). *Let $M^<$ be a $<$-matrix. Every sequence of rule applications starting with the tuple $\langle \epsilon, M^<, \epsilon \rangle$ terminates.*

Theorem 3 (Soundness). *If there exists a preferential connection proof for $\langle \epsilon, M^<, \epsilon \rangle$ with an admissible θ-substitution, then there is a multiplicity μ s.t. every path through $M_\mu^<$ contains a θ-complementary connection.*

Theorem 4 (Completeness). *If every path through $M_\mu^<$ contains a θ-complementary connection, then there is a preferential connection proof for $\langle \epsilon, M^<, \epsilon \rangle$.*

4 Example of Proof

In Example 1, assume we want to know whether Hermione is a Muggle, but not a typical one. We can express this query as $(\mathsf{Muggle} \sqcap \neg\bullet\mathsf{Muggle})(\mathsf{hermione})$.

Let us transform the query into TNF. As it is a concept assertion, it is replaced by $\mathsf{N}_1(\mathsf{hermione})$, adding $\mathsf{Muggle} \sqcap \neg\bullet\mathsf{Muggle} \sqsubseteq \mathsf{N}_1$ in the knowledge base. Next, we rewrite this new axiom, based on the fact that $A \sqcap N \sqsubseteq C, B \sqsubseteq N$ is a satisfiability-preserving rewriting of $A \sqcap B \sqsubseteq C$. We end up with the following knowledge base:

$$\mathcal{T} = \left\{ \begin{array}{c} \bullet\mathsf{Muggle} \sqsubseteq \neg\mathsf{Wizard}, \\ \exists\mathsf{casts}.\mathsf{Spell} \sqsubseteq \mathsf{Wizard}, \\ \mathsf{Muggle} \sqcap \mathsf{N}_2 \sqsubseteq \mathsf{N}_1, \\ \neg\bullet\mathsf{Muggle} \sqsubseteq \mathsf{N}_2 \end{array} \right\} \qquad \mathcal{A} = \left\{ \begin{array}{c} \mathsf{Muggle}(\mathsf{hermione}), \\ \mathsf{Spell}(\mathsf{oculusReparo}), \\ \mathsf{casts}(\mathsf{hermione}, \mathsf{oculusReparo}) \end{array} \right\}$$

The query $\mathsf{N}_1(\mathsf{hermione})$ is a logical consequence of $\mathcal{K}$ if the matrix $M(\neg\mathcal{K}) \cup \{\{\mathsf{N}_1(\mathsf{hermione})\}\}$ is valid. The resulting matrix and its partial order are below:

$$\begin{bmatrix} \mathsf{M}(x_0) & \mathsf{M}(x_1) & \mathsf{c}(x_2, y_1) & & & & \mathsf{M}(x_3) & \neg\mathsf{M}(x_4) & \mathsf{M}(y_3) & \\ \neg\mathsf{M}(e) & \mathsf{M}(y_0) & \mathsf{S}(y_1) & \neg\mathsf{M}(\mathsf{h}) & \neg\mathsf{S}(\mathsf{r}) & \neg\mathsf{c}(\mathsf{h},\mathsf{r}) & \mathsf{N}_2(x_3) & \neg\mathsf{N}_2(x_4) & \neg\mathsf{N}_2(x_5) & \mathsf{N}_1(\mathsf{h}) \\ \mathsf{W}(x_0) & \mathsf{W}(x_1) & \neg\mathsf{W}(x_2) & & & & \neg\mathsf{N}_1(x_3) & & & \end{bmatrix}$$

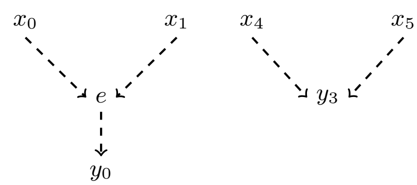

Fig. 2. Matrix representation and its strict partial order. The graphical matrix displays clauses as columns in the same order as shown in $\mathcal{K}$. Gaps may happen in the matrix since the clauses can have different lengths.

With the relation at hand, we are able to restrict the θ-substitutions. For instance, in the example in Figure 2, we cannot do a θ-substitution for e and x_0 because $(e, x_0) \in <$, and it would violate the preference condition. However, we can do a θ-substitution for e and x_4. When we carry out this last substitution, the lattice of the preference relation is modified during the construction of the path to accommodate the θ-substitution, as shown in Figure 3.

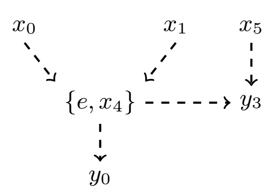

Fig. 3. Strict partial order after the substitution $\theta(e) = \theta(x_4)$.

A proof in this method can be built in two equivalent ways: a simpler matrix with connections and a proof in the calculus. In Figure 4, we show the proof using the former: the matrix, the θ-substitution, and the final partial order. In the figure, connections have been numbered to represent the order in which they were built.

The same proof using the calculus is shown in Figure 5.

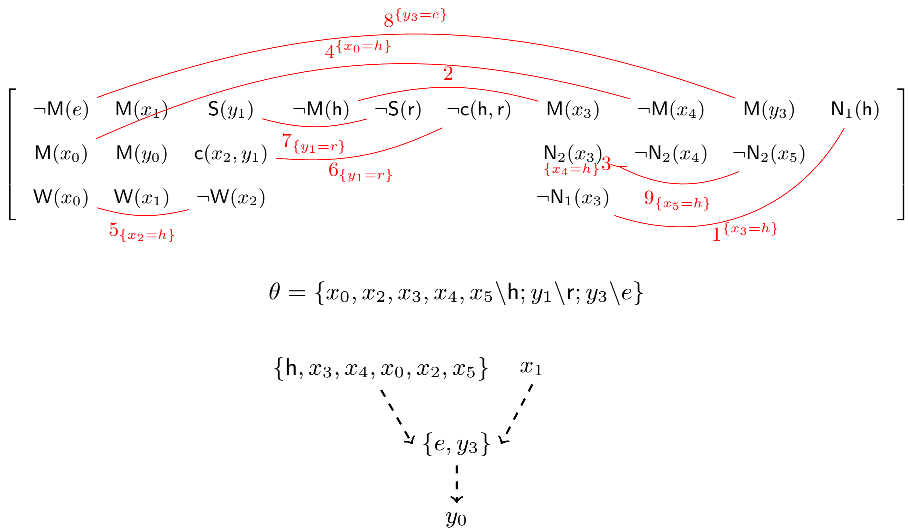

Fig. 4. Connection proof in matrix style, with θ-substitutions and preference relation. Some literals have been rearranged for a better visualization of the connections.

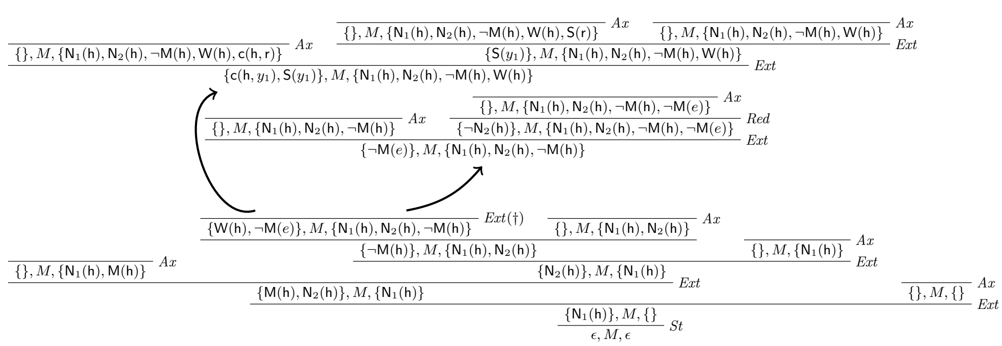

Fig. 5. Connection proof in calculus style. The arrows copy the two subproofs generated after the extension rule $Ext(\dagger)$. It must also include the θ-substitution and partial order of Figure 4 (not duplicated here for the sake of space).

5 Concluding Remarks

In this work, we have defined a connection method for $\mathcal{ALC}^{\bullet}$, a commonly considered fragment of the defeasible description logic $\mathcal{ALCH}^{\bullet}$ [31]. The calculus extends the one by Freitas and Otten, $\mathcal{ALC}$ θ-CM [21], in a number of aspects: (i) it complies with the preferential-DL semantics developed by Britz et al. and by Giordano et al. and which is widely assumed in the literature on reasoning with defeasible ontologies; (ii) it relies on a tailor-made normal form we introduced to cater for typicality in concepts, and (iii) it includes a new internal structure, namely a preference relation on objects, which is built during the proof process and allows for an elegant handling of connections involving typicality.

As already alluded to in the introduction, the work here reported makes the first steps in the study of connection methods as viable alternatives for reasoning with defeasible ontologies. It is therefore part of a broader long-term agenda. Immediate next steps stemming from the present work are: (i) extending the method to the full language of $\mathcal{ALCH}^{\bullet}$, in particular to handle typicality of role names and in role hierarchies, and (ii) endowing Raccoon with the ability to reason over defeasible extensions of $\mathcal{ALC}$.

The reader conversant with preferential reasoning would have noticed that in this work we assume preferential entailment, which is a Tarskian notion of consequence and therefore monotonic. As pointed out in the literature on nonmonotonic reasoning, preferential entailment is not always enough for reasoning defeasibly with exceptions. Stronger, more robust, forms of entailment are often called for and one particular definition thereof, namely the *rational closure* of a defeasible ontology has been thoroughly investigated in the context of defeasible $\mathcal{ALC}$ [8,18,28]. Nevertheless, a case has been made for sticking to preferential reasoning in some contexts [26] or for investigating weaker forms of rationality [7,23], which suggests the debate around rational closure being the baseline for defeasible reasoning remains, in a sense, open.

In any case, the aforementioned approaches to the computation of the rational closure of a defeasible knowledge base rely on a number of calls to a classical (monotonic) reasoner and therefore the availability of a method for preferential reasoning in $\mathcal{ALC}^{\bullet}$ as the one we propose here is an important step in the investigation of stronger forms of entailment in the presence of typicality operators.

The limitations of adopting a single preference ordering in modelling object typicality have already been pointed out by Britz and Varzinczak [15]. They introduce a notion of context in a multi-preference semantics making it possible for some objects to be more typical than others w.r.t. a context, but less typical w.r.t. a different one. Part of our research agenda is therefore to extend the method here proposed to deal with multiple preference relations. The tableau system of Britz and Varzinczak can serve as a springboard with which to investigate such an extension to contextual defeasible reasoning.

Acknowledgments

This study has been financed in part by the Coordenação de Aperfeiçoamento de Pessoal de Nível Superior - Brasil (CAPES) - Finance Code 001. It also has been supported in part by the project *Reconciling Description Logics and Non-Monotonic Reasoning in the Legal Domain* (PRC CNRS–FACEPE, France–Brazil). Fred Freitas is partially supported by a Research Productivity grant provided by the Brazilian funding agency National Council for Scientific and Technological Development – CNPq.

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