# PTL: A Propositional Typicality Logic*

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**Abstract.** We introduce Propositional Typicality Logic (PTL), a logic for reasoning about typicality. We do so by enriching classical propositional logic with a typicality operator of which the intuition is to capture the most typical (or normal) situations in which a formula holds. The semantics is in terms of ranked models as studied in KLM-style preferential reasoning. This allows us to show that rational consequence relations can be embedded in our logic. Moreover we show that we can define consequence relations on the language of PTL itself, thereby moving beyond the propositional setting. Building on the existing link between propositional rational consequence and belief revision, we show that the same correspondence holds for rational consequence and belief revision on PTL. We investigate entailment for PTL, and propose two appropriate notions thereof.

**Keywords:** Nonmonotonic reasoning; typicality; belief revision; rationality

## 1 Introduction and Motivation

The preferential and rational consequence relations first studied by Lehmann and colleagues in the 90's play a central role in nonmonotonic reasoning [13, 14]. This has been the case due to at least three main reasons. Firstly, they are based on semantic constructions that are elegant and neat. Secondly, they provide the foundation for the determination of the important notion of entailment in this context. Finally they also offer an alternative perspective on belief change [9].

A curious aspect of such consequence relations (and corresponding belief revision constructions) is that they are crucially, albeit tacitly, based on a notion of *typicality*. However, in the corresponding underlying language it is not possible to refer directly to such a notion. In this paper, we fill this gap with the introduction of an explicit operator to talk about typicality. Intuitively, our new syntactic construction allows us to single out those most typical situations in which a formula holds. The result is a more expressive language allowing us, for instance, to make statements of the form "the most typical $\alpha$s are most typical $\beta$s", which is not possible in the aforementioned frameworks.

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The remainder of the paper is structured as follows: After some preliminaries (Section 2), we define and investigate PTL, a propositional typicality logic extending propositional logic (Section 3). The semantics of PTL is in terms of ranked models as studied in the literature on preferential reasoning. This allows us to embed propositional KLM-style consequence relations in our new language. In Section 4 we show that, although the addition of the typicality operator increases the expressivity of the logic, the nesting of the typicality does not. In Section 5 we investigate the link between AGM belief revision and PTL. We show that propositional AGM belief revision can be expressed in terms of typicality, and also that it can be lifted to a version of revision on PTL. We then move to an investigation of rational consequence relations in terms of PTL (Section 6). We show that propositional rational consequence can be expressed in PTL, that it can be extended to PTL in terms of PTL itself, and that the propositional connection between rational consequence and revision carries over to PTL. In Section 7 we raise the question of what an appropriate notion of entailment for PTL is, and propose at least two candidates. After a discussion of and comparison with related work (Section 8), we conclude with a summary of the contributions and directions for further investigation.

## 2 Preliminaries

We work in a propositional language over a finite set of *atoms* $\mathcal{P}$, denoted by $p, q, \ldots$ (In later sections we adopt a richer language.) Propositional formulas (and in later sections, formulas of the richer language) are denoted by $\alpha, \beta, \ldots$, and are recursively defined in the usual way: $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha$. The other truth-functional connectives are defined in terms of $\neg$ and $\wedge$ in the usual way. We use $\top$ as an abbreviation for $p \vee \neg p$, and $\bot$ for $p \wedge \neg p$, for some $p \in \mathcal{P}$. With $\mathcal{L}$ we denote the set of all propositional formulas.

We denote by $\mathscr{U}$ the set of all *valuations* $v : \mathcal{P} \longrightarrow \{0, 1\}$. Satisfaction of $\alpha \in \mathcal{L}$ by $v \in \mathscr{U}$ is defined in the usual truth-functional way. With $Mod(\alpha)$ we denote the set of all valuations satisfying $\alpha$. Given sentences $\alpha$ and $\beta$, $\alpha \models \beta$ ($\alpha$ entails $\beta$) means $Mod(\alpha) \subseteq Mod(\beta)$. We extend the notions of $Mod(\cdot)$ and entailment to knowledge bases in the usual way: for a finite $\mathcal{K} \subseteq \mathcal{L}$, $Mod(\mathcal{K})$ is the set of all valuations satisfying every formula in $\mathcal{K}$, and $\mathcal{K} \models \alpha$ if and only if $Mod(\mathcal{K}) \subseteq Mod(\alpha)$.

A propositional defeasible consequence relation $\mathrel{|\!\sim}$ is defined as a binary relation on the formulas of the underlying (finitely generated) propositional logic. $\mathrel{|\!\sim}$ is said to be *preferential* if it satisfies the following set of properties [13]:

$$\text{(Ref) } \alpha \mathrel{|\!\sim} \alpha \qquad \text{(LLE) } \frac{\models \alpha \leftrightarrow \beta, \ \alpha \mathrel{|\!\sim} \gamma}{\beta \mathrel{|\!\sim} \gamma} \qquad \text{(And) } \frac{\alpha \mathrel{|\!\sim} \beta, \ \alpha \mathrel{|\!\sim} \gamma}{\alpha \mathrel{|\!\sim} \beta \wedge \gamma}$$

$$\text{(Or) } \frac{\alpha \mathrel{|\!\sim} \gamma, \ \beta \mathrel{|\!\sim} \gamma}{\alpha \vee \beta \mathrel{|\!\sim} \gamma} \qquad \text{(RW) } \frac{\alpha \mathrel{|\!\sim} \beta, \ \models \beta \rightarrow \gamma}{\alpha \mathrel{|\!\sim} \gamma} \qquad \text{(CM) } \frac{\alpha \mathrel{|\!\sim} \beta, \ \alpha \mathrel{|\!\sim} \gamma}{\alpha \wedge \beta \mathrel{|\!\sim} \gamma}$$

If, in addition to the properties of preferential consequence, $\mathrel{|\!\sim}$ also satisfies the following Rational Monotonicity property, it is said to be a *rational* consequence relation [14]:

$$\text{(RM) } \frac{\alpha \mathrel{|\!\sim} \beta, \ \alpha \mathrel{|\!\not\sim} \neg\gamma}{\alpha \wedge \gamma \mathrel{|\!\sim} \beta}$$

The semantics of (propositional) rational consequence is in terms of *ranked* models. These are partially ordered structures in which the ordering is *modular*.

**Definition 1.** *Given a set $S$, $\prec \subseteq S \times S$ is modular if and only if there is a ranking function $rk : S \longrightarrow \mathbb{N}$ such that for every $s, s' \in S$, $s \prec s'$ if and only if $rk(s) < rk(s')$.*

**Definition 2.** *A ranked model $\mathscr{R}$ is a pair $\langle \mathcal{V}, \prec \rangle$, where $\mathcal{V} \subseteq \mathscr{U}$ and $\prec \subseteq \mathcal{V} \times \mathcal{V}$ is a modular order over $\mathcal{V}$.*[3]

**Definition 3.** *Let $\alpha \in \mathcal{L}$ and let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be a ranked model. With $[\![\alpha]\!]$ we denote the set of valuations satisfying $\alpha$ in $\mathscr{R}$, defined as follows:*

$$[\![p]\!] := \{v \in \mathcal{V} \mid v(p) = 1\}, \;\; [\![\neg\alpha]\!] := \mathcal{V} \setminus [\![\alpha]\!], \;\; [\![\alpha \wedge \beta]\!] := [\![\alpha]\!] \cap [\![\beta]\!]$$

Given a ranked model $\mathscr{R}$, the intuition is that valuations lower down in the ordering are more preferred than those higher up. Hence, a pair $(\alpha, \beta)$ is in the consequence relation defined by $\mathscr{R}$ (denoted as $\alpha \mid\!\sim_{\mathscr{R}} \beta$) if and only if $\min_{\prec}[\![\alpha]\!] \subseteq [\![\beta]\!]$, i.e., the most preferred (with respect to $\prec$) $\alpha$-valuations are also $\beta$-valuations.

Lehmann and Magidor provided a representation result for the propositional case, establishing that a defeasible consequence relation $\mid\!\sim$ on $\mathcal{L}$ is rational if and only if it is defined by some ranked model [14, 9].

## 3 Propositional Typicality Logic

We introduce now a propositional typicality logic, called PTL, which extends propositional logic with a typicality operator $\overline{\phantom{x}}$ (read 'bar'). The language of PTL, denoted by $\overline{\mathcal{L}}$, is recursively defined by: $\alpha ::= p \mid \neg\alpha \mid \alpha \wedge \alpha \mid \overline{\alpha}$. (As before, the other connectives are defined in terms of $\neg$ and $\wedge$, and $\top$ and $\bot$ are abbreviations.) Intuitively, $\overline{\alpha}$ is understood to refer to the typical situations in which $\alpha$ holds. The semantics is in terms of ranked models and we extend the notion of satisfaction from Definition 3 as follows:

**Definition 4.** *Let $\alpha \in \overline{\mathcal{L}}$ and let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$. Then $[\![\overline{\alpha}]\!] := \min_{\prec}[\![\alpha]\!]$.*

Given $\alpha \in \overline{\mathcal{L}}$ and $\mathscr{R}$ a ranked model, we say that $\alpha$ is *true* in $\mathscr{R}$ (denoted as $\mathscr{R} \Vdash \alpha$) if $[\![\alpha]\!] = \mathcal{V}$. For $\mathcal{K} \subseteq \overline{\mathcal{L}}$, $\mathscr{R} \Vdash \mathcal{K}$ if $\mathscr{R} \Vdash \alpha$ for every $\alpha \in \mathcal{K}$. $\alpha$ is *valid*, denoted as $\models \alpha$, if $\mathscr{R} \Vdash \alpha$ for every ranked model $\mathscr{R}$.

Note that for every ranked model $\mathscr{R}$ and $\alpha \in \overline{\mathcal{L}}$, there is a $\beta \in \mathcal{L}$ (i.e., a propositional formula) such that $\mathscr{R} \Vdash \alpha \leftrightarrow \beta$. That is to say, given $\mathscr{R}$, every $\alpha$ can be expressed as a propositional formula ($\beta$) in $\mathscr{R}$. Of course, this does not mean that propositional logic is as expressive as PTL, since the formula $\beta$ used to express $\alpha$ in the ranked model $\mathscr{R}$ depends on $\mathscr{R}$. Rather, the relationship between PTL and propositional logic is similar to the relationship between modal logic and propositional logic in the sense that both modal logic and PTL add to propositional logic an operator that is not truth-functional. (In Section 8 we discuss in more detail the relationship between PTL and modal logic.)

Next is a property which shows that if $\mathscr{R} \Vdash \overline{\alpha}$, then $\mathscr{R}$ consists of only $\alpha$-worlds in which all worlds are incomparable (alias equally preferred) according to $\prec$.

[3] This is not Lehmann and Magidor's [14] original definition of ranked models but a characterization of rational consequence can be given in terms of ranked models as we present here [9].

**Proposition 1.** *Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$. Then (1) $\prec = \emptyset$ iff there is an $\alpha \in \overline{\mathcal{L}}$ such that $\mathscr{R} \Vdash \overline{\alpha}$; (2) for every $\alpha \in \overline{\mathcal{L}}$, $\mathscr{R} \Vdash \overline{\alpha}$ iff for every $\beta \in \mathcal{L}$ such that $\mathscr{R} \Vdash \alpha \rightarrow \beta$, $\mathscr{R} \Vdash \overline{\beta}$.*

One of the consequences of this result is that if $\overline{\alpha}$ is true in a ranked model, then so is $\alpha$ (but the converse, of course, does not hold).

Another useful property of typicality is that it allows us to express (propositional) rational consequence, as defined in Section 2.

**Proposition 2.** *For $\alpha, \beta \in \mathcal{L}$, $\alpha \mathrel{|\!\sim}_{\mathscr{R}} \beta$ if and only if $\mathscr{R} \Vdash \overline{\alpha} \rightarrow \beta$.*

Proposition 2 shows that the introduction of typicality into the object language allows us to express rational consequence. This forms part of our argument to show that our semantics for typicality is the correct one, but it does not provide a justification for introducing all the additional expressivity obtained from typicality. To provide such a justification we turn to the notion of *defeasible incompatibility*. Intuitively, $\alpha$ and $\beta$ are said to be *incompatible* if they are contradictory. Therefore with an appropriate definition of defeasible incompatibility we should be able to capture the idea of $\alpha$ and $\beta$ being defeasibly incompatible. There seems to be at least four different ways of expressing defeasible incompatibility, none of which are equivalent (with respect to ranked models), but all of which would be propositionally equivalent if the typicality operator were removed: (*i*) $(\overline{\alpha} \rightarrow \neg\beta) \wedge (\overline{\beta} \rightarrow \neg\alpha)$; (*ii*) $\overline{\top} \rightarrow \neg(\alpha \wedge \beta)$; (*iii*) $\overline{\neg(\alpha \wedge \beta)}$, and (*iv*) $\neg(\overline{\alpha} \wedge \overline{\beta})$. Observe that (*i*) can be expressed as two $\mathrel{|\!\sim}$-statements ($\alpha \mathrel{|\!\sim} \neg\beta$ and $\beta \mathrel{|\!\sim} \neg\alpha$), that (*ii*) can be expressed in terms of $\mathrel{|\!\sim}$, but that (*iii*) and (*iv*) cannot.

Furthermore, although it may be useful to be able to express all four of these options, our contention is that the notion of defeasible incompatibility is correctly captured by option (*iv*), one of the options that *cannot* be expressed in terms of $\mathrel{|\!\sim}$. To see why, note firstly that option (*i*) is ruled out because it is too strong. It expressly forbids typical $\alpha$-situations to be $\beta$-situations (and forbids typical $\beta$-situations to be $\alpha$-situations). We could consider weakening it so that typical $\alpha$-situations are only forbidden to be *typical* $\beta$-situations (and similarly with the roles of $\alpha$ and $\beta$ reversed), i.e., to $(\overline{\alpha} \rightarrow \neg\overline{\beta}) \wedge (\overline{\beta} \rightarrow \neg\overline{\alpha})$. That looks reasonable indeed, but it is easy to see that this statement is equivalent to each of its two conjuncts $\overline{\alpha} \rightarrow \neg\overline{\beta}$ and $\overline{\beta} \rightarrow \neg\overline{\alpha}$, and also to option (*iv*). To see why option (*ii*) does not fit the bill either, it is best to consider its representation in terms of $\mathrel{|\!\sim}$: $\top \mathrel{|\!\sim} \neg(\alpha \wedge \beta)$. From this we do not always get $\gamma \mathrel{|\!\sim} \neg(\alpha \wedge \beta)$. Thus, in a sense, option (*ii*) is too weak since it ignores, for the most part, the context in which defeasible incompatibility is supposed to hold. For option (*iii*), from Proposition 1 it follows that if $\overline{\neg(\alpha \wedge \beta)}$ holds, then so does $\neg(\alpha \wedge \beta)$, which is clearly too strong. Finally, observe that option (*iv*) is interpreted to mean that the most typical $\alpha$-situations and the most typical $\beta$-situations are incompatible, which corresponds best to the informal notion of defeasible incompatibility. In summary then, it seems that to express defeasible incompatibility correctly, it is necessary to go beyond rational consequence, but sufficient to introduce typicality into the object language.

Finally, observe that Proposition 2 shows that rational consequence for *propositional logic* can be expressed in PTL. In Section 6 we shall see that it is also possible to express, in PTL itself, the extended notion of rational consequence *for the language $\overline{\mathcal{L}}$*.

## 4 Typicality Unraveled

In the previous section we have argued for the need to include typicality explicitly in the object language. The observant reader would have noticed that $\overline{\mathcal{L}}$ allows for the arbitrary (finite) nesting of the typicality operator. An important point to consider is whether this much expressivity is needed, and whether it is not perhaps sufficient to restrict the language to non-nested applications of typicality.

In this section we show that once typicality is added to the language, nesting does not increase the expressivity any further, provided that we are allowed to add new propositional atoms. We shall thus be working with languages in which the set of propositional atoms $\mathcal{P}$ may vary, and more specifically, with languages with respect to a given knowledge base. So, given a knowledge base $\mathcal{K}$, we denote by $\mathcal{P}^{\mathcal{K}}$ the set of propositional atoms occurring in $\mathcal{K}$. Furthermore, by a *ranked model on* $\mathcal{P}^{\mathcal{K}}$ we mean a ranked model built up using only the propositional atoms occurring in $\mathcal{P}^{\mathcal{K}}$.

Now, given any $\mathcal{K} \subseteq \overline{\mathcal{L}}$ we: (*i*) Show how to transform every $\beta \in \overline{\mathcal{L}}$ into a formula $\widehat{\beta}$ containing no nested instances of the bar operator (and therefore also how to transform $\mathcal{K}$ into a knowledge base $\widehat{\mathcal{K}}$, containing no nested instances of the bar operator); (*ii*) Show how to construct an auxiliary set of formulas $\widehat{E}$, containing no nested instances of the bar operator, regulating the behavior of the newly introduced propositional atoms, and (*iii*) Show how to transform every ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ into its "appropriate representative" $\widehat{\mathscr{R}}$ on $\mathcal{P}^{\widehat{\mathcal{K}}}$ such that, for every $\beta \in \overline{\mathcal{L}}$, $\beta$ is true in $\mathscr{R}$ if and only if $\widehat{\beta}$ is true in $\widehat{\mathscr{R}}$. Using these constructions we show that $\widehat{\mathcal{K}} \cup \widehat{E}$ is the non-nested version of $\mathcal{K}$ in the sense that the ranked models in which $\widehat{\mathcal{K}} \cup \widehat{E}$ are true are precisely the "appropriate representatives" of the ranked models in which $\mathcal{K}$ is true.

To be more precise, let $\mathcal{K} \subseteq \overline{\mathcal{L}}$, let $S^{\mathcal{K}}$ denote all subformulas of $\mathcal{K}$, and let $B^{\mathcal{K}} = \{\overline{\alpha} \in S^{\mathcal{K}} \mid \alpha \in \mathcal{L}\}$. So $B^{\mathcal{K}}$ contains all occurrences of subformulas in $\mathcal{K}$ containing a single bar. Informally, the idea is to substitute (all occurrences of) every element $\overline{\alpha}$ of $B^{\mathcal{K}}$ with a new atom $p^{\overline{\alpha}}$, and to require that $p^{\overline{\alpha}}$ be equivalent to $\overline{\alpha}$. In doing so we reduce the level of nesting in $\mathcal{K}$ by a factor of 1. Now, let $E^{\mathcal{K}} = \{p^{\overline{\alpha}} \leftrightarrow \overline{\alpha} \mid \overline{\alpha} \in B^{\mathcal{K}}\}$, and for every $\beta \in \overline{\mathcal{L}}$, let $\beta^{\mathcal{K}}$ be obtained from $\beta$ by the simultaneous substitution in $\beta$ of (every occurrence of) every $\overline{\alpha} \in B^{\mathcal{K}}$ by $p^{\overline{\alpha}}$ (observe that $\beta^{\mathcal{K}} = \beta$ if $\beta$ is a propositional formula). We refer to $\beta^{\mathcal{K}}$ as the $\mathcal{K}$*-transform of* $\beta$. Also, let $\overline{\mathcal{K}} = \{\beta^{\mathcal{K}} \mid \beta \in \mathcal{K}\}$. The idea is that $\overline{\mathcal{K}} \cup E^{\mathcal{K}}$ is a version of $\mathcal{K}$ with one fewer level of nesting.

*Example 1.* Let $\mathcal{K} = \{\overline{p \wedge q} \to r, \overline{\overline{p} \vee r}, \overline{p \wedge \overline{q \vee \overline{r}}}\}$. Then $B^{\mathcal{K}} = \{\overline{p \wedge q}, \overline{p}, \overline{r}\}$ and $E^{\mathcal{K}} = \{p^{\overline{p \wedge q}} \leftrightarrow \overline{p \wedge q}, p^{\overline{p}} \leftrightarrow \overline{p}, p^{\overline{r}} \leftrightarrow \overline{r}\}$. Now $(\overline{p \wedge q} \to r)^{\mathcal{K}} = p^{\overline{p \wedge q}} \to r$, $(\overline{\overline{p} \vee r})^{\mathcal{K}} = \overline{p^{\overline{p}} \vee r}$, $(\overline{p \wedge \overline{q \vee \overline{r}}})^{\mathcal{K}} = \overline{p \wedge \overline{q \vee p^{\overline{r}}}}$. Hence $\overline{\mathcal{K}} = \{p^{\overline{p \wedge q}} \to r, \overline{p^{\overline{p}} \vee r}, \overline{p \wedge \overline{q \vee p^{\overline{r}}}}\}$. Observe that $\mathcal{K}$ has a nesting level of 3, while $\overline{\mathcal{K}}$ has a nesting level of 2.

Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be a ranked model on $\mathcal{P}^{\mathcal{K}}$. We define $\overline{\mathscr{R}} = (\overline{\mathcal{V}}, \overline{\prec})$ on $\mathcal{P}^{\overline{\mathcal{K}}}$ as follows: for all $v \in \mathcal{V}$, let $\overline{v}$ be a valuation on $\mathcal{P}^{\overline{\mathcal{K}}}$ such that (*i*) $\overline{v}(p) = v(p)$ for every $p \in \mathcal{P}^{\mathcal{K}}$, and (*ii*) for every $p^{\overline{\alpha}} \in (\mathcal{P}^{\overline{\mathcal{K}}} \setminus \mathcal{P}^{\mathcal{K}})$, $\overline{v}(p^{\overline{\alpha}}) = 1$ if and only if $v \in [\![\overline{\alpha}]\!]$ in $\mathscr{R}$. And for all $\overline{v}, \overline{v'} \in \overline{\mathcal{V}}$, $\overline{v} \,\overline{\prec}\, \overline{v'}$ if and only if $v \prec v'$. So $\overline{\mathscr{R}}$ is an extended version of $\mathscr{R}$ with every valuation $v$ in $\mathscr{R}$ replaced with an extended valuation $\overline{v}$ in which the truth values of atoms occurring in $v$ remain unchanged, and the truth values of the new atoms are constrained by the requirement that every $p^{\overline{\alpha}}$ be equivalent to $\overline{\alpha}$ (for $\overline{\alpha} \in B^{\mathcal{K}}$). We refer to $\overline{\mathscr{R}}$ as the $\mathcal{K}$*-extended version of* $\mathscr{R}$. From this we obtain the following result.

**Proposition 3.** *For every ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$, $\overline{\mathscr{R}}$ satisfies $E^{\mathcal{K}}$. Conversely, a ranked model $\mathscr{R}'$ on $\mathcal{P}^{\overline{\mathcal{K}}}$ satisfies $E^{\mathcal{K}}$ if and only if there is a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ such that $\overline{\mathscr{R}} = \mathscr{R}'$. Furthermore, let $\mathscr{R}$ be a ranked model on $\mathcal{P}^{\mathcal{K}}$. Then $\mathscr{R}$ satisfies $\mathcal{K}$ if and only if $\overline{\mathscr{R}} \Vdash \overline{\mathcal{K}}$. Also, for all $\beta \in \overline{\mathcal{L}}$, $\mathscr{R} \Vdash \beta$ if and only if $\overline{\mathscr{R}} \Vdash \overline{\beta}$.*

The proposition above shows that the $\mathcal{K}$-extended version of a ranked model $\mathscr{R}$ is the only "appropriate representative" of $\mathscr{R}$ in the class of ranked models based on the extended language of $\overline{\mathcal{K}}$. Also, the $\mathcal{K}$-extended versions of the ranked models based on the language of $\mathcal{K}$ are the only ones satisfying $E^{\mathcal{K}}$.

As mentioned above, the move from $\mathcal{K}$ to $\overline{\mathcal{K}}$ ensures that we can reduce the level of nesting of $^{-}$ by a factor of 1. To arrive at a set $\widehat{\mathcal{K}}$ not containing any nested occurrences of $^{-}$ we just need to iterate the transform process a sufficient number of times. So, we define $\widehat{\mathcal{K}}$ as follows: Let $\mathcal{K}_0 = \mathcal{K}$, and for $i > 0$, let $B_i = B^{\mathcal{K}_{i-1}}$, $\mathcal{K}_i = \overline{\mathcal{K}_{i-1}}$, and let $n = \min_{<}\{i \mid B_{i+1} = \emptyset\}$. We then let $\widehat{\mathcal{K}} = \mathcal{K}_n$. So for every $i = 1, \ldots, n$, $\mathcal{K}_i$ has one fewer level of nesting of $^{-}$ than $\mathcal{K}_{i-1}$ until we get to $\mathcal{K}_n = \widehat{\mathcal{K}}$, which has no nested occurrences of $^{-}$. Similarly, for every $\beta \in \overline{\mathcal{L}}$, we define $\widehat{\beta}$ as follows: Let $\beta_0 = \beta$, for $i = 1, \ldots, n$, let $\beta_i = \beta^{\mathcal{K}_{i-1}}$, and let $\widehat{\beta} = \beta_n$. We refer to $\widehat{\beta}$ as the *full* $\mathcal{K}$*-transform of* $\beta$. In a similar vein, we let $\widehat{E} = \bigcup_{i=0}^{i=n-1} E^{\mathcal{K}_i}$.

*Example 2.* Continuing Example 1, let $\mathcal{K}_0 = \mathcal{K}$. Then $B_1 = B^{\mathcal{K}_0} = B^{\mathcal{K}}$, and $\mathcal{K}_1 = \overline{\mathcal{K}}$ with $E_0 = E^{\mathcal{K}}$; $B_2 = B^{\mathcal{K}_1} = \{\overline{q \vee p^{\overline{r}}}\}$, and $E_1 = \{p^{\overline{q \vee p^{\overline{r}}}} \leftrightarrow \overline{q \vee p^{\overline{r}}}\}$. Now $\mathcal{K}_2 = \overline{\mathcal{K}_1} = \overline{\overline{\mathcal{K}}} = \{p^{\overline{p \wedge q}} \to r, \overline{p^{\overline{p}} \vee r}, \overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}\}$. In the 2nd iteration, $B_3 = B^{\mathcal{K}_2} = \{\overline{p^{\overline{p}} \vee r}, \overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}\}$ with $E_2 = \{p^{\overline{p^{\overline{p}} \vee r}} \leftrightarrow \overline{p^{\overline{p}} \vee r}, p^{\overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}} \leftrightarrow \overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}\}$. Then $\mathcal{K}_3 = \overline{\mathcal{K}_2} = \{p^{\overline{p \wedge q}} \to r, p^{\overline{p^{\overline{p}} \vee r}}, p^{\overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}}\}$. In the next iteration, $B_4 = \emptyset$. Hence $n = 3$, and

$$\widehat{\mathcal{K}} = \left\{ \begin{array}{c} p^{\overline{p \wedge q}} \to r, p^{\overline{p^{\overline{p}} \vee r}}, \\ p^{\overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}} \end{array} \right\}, \widehat{E} = \left\{ \begin{array}{c} p^{\overline{p \wedge q}} \leftrightarrow \overline{p \wedge q}, p^{\overline{p}} \leftrightarrow \overline{p}, p^{\overline{r}} \leftrightarrow \overline{r}, p^{\overline{q \vee p^{\overline{r}}}} \leftrightarrow \overline{q \vee p^{\overline{r}}}, \\ p^{\overline{p^{\overline{p}} \vee r}} \leftrightarrow \overline{p^{\overline{p}} \vee r}, p^{\overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}}} \leftrightarrow \overline{p \wedge p^{\overline{q \vee p^{\overline{r}}}}} \end{array} \right\}$$

Finally, for any ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$, we define its *full* $\mathcal{K}$*-extended version* $\widehat{\mathscr{R}}$ as follows: Let $\mathscr{R}_0 = \mathscr{R}$, and for $i = 1, \ldots, n$, $\mathscr{R}_i = \overline{\mathscr{R}_{i-1}}$. Then we let $\widehat{\mathscr{R}} = \mathscr{R}_n$.

Using Proposition 3 we then obtain the result we require.

**Theorem 1.** *For every $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$, its full $\mathcal{K}$-extended version $\widehat{\mathscr{R}}$ satisfies $\widehat{E}$. Conversely, a ranked model $\mathscr{R}'$ on $\mathcal{P}^{\widehat{\mathcal{K}}}$ satisfies $\widehat{E}$ if and only if there is a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$ such that $\mathscr{R}' = \widehat{\mathscr{R}}$. Furthermore, let $\mathscr{R}$ be a ranked model $\mathscr{R}$ on $\mathcal{P}^{\mathcal{K}}$. Then $\mathscr{R} \Vdash \mathcal{K}$ if and only if $\widehat{\mathscr{R}} \Vdash \widehat{\mathcal{K}}$. Also, for all $\beta \in \overline{\mathcal{L}}$, $\mathscr{R} \Vdash \beta$ if and only if $\widehat{\mathscr{R}} \Vdash \widehat{\beta}$.*

## 5 Belief Revision and Typicality

Given the well-known link between propositional rational consequence and AGM belief revision [1], as developed by Gärdenfors and Makinson [9], it is perhaps not surprising that propositional AGM belief revision can be expressed using the typicality operator. In this section we make this claim precise. The formal representation of propositional AGM revision we provide below is based on that of Katsuno and Mendelzon [12].

The starting point is to fix a non-empty subset $\mathcal{V}$ of $\mathscr{U}$ (as done by Kraus et al. [13]), and to assume that everything is done within the context of $\mathcal{V}$. In that sense, $\mathcal{V}$ becomes the set of *all* valuations available to us. This is slightly more general than the Katsuno-Mendelzon framework which assumes $\mathcal{V}$ to be equal to $\mathscr{U}$, but is a special case of the original AGM approach. To reflect this restriction, we use $Mod_{\mathcal{V}}(\alpha)$ to denote the set $Mod(\alpha) \cap \mathcal{V}$. In the same vein, in the postulates below, validity is understood to be modulo $\mathcal{V}$. That is, for $\alpha \in \mathcal{L}$ we let $\models \alpha$ if and only if $Mod_{\mathcal{V}}(\alpha) = \mathcal{V}$.

Next, we fix a knowledge base $\kappa \in \mathcal{L}$ (i.e., represented as a propositional formula) such that $Mod_{\mathcal{V}}(\kappa) \neq \emptyset$. A revision operator $\circ$ on $\mathcal{L}$ for $\kappa$ is a function from $\mathcal{L}$ to $\mathcal{L}$. Intuitively, $\kappa \circ \alpha$ is the result of revising $\kappa$ by $\alpha$ (clearly the models of $\kappa \circ \alpha$ should be in $\mathcal{V}$). An AGM revision operator $\circ$ on $\mathcal{L}$ for $\kappa$ is a revision operator on $\mathcal{L}$ for $\kappa$ which satisfies the following six properties:

**(R1)** $\models (\kappa \circ \alpha) \rightarrow \alpha$
**(R2)** If $\not\models \neg(\kappa \wedge \alpha)$, then $\models (\kappa \circ \alpha) \leftrightarrow (\kappa \wedge \alpha)$
**(R3)** If $\not\models \neg\alpha$, then $\not\models \neg(\kappa \circ \alpha)$
**(R4)** If $\models \kappa_1 \leftrightarrow \kappa_2$ and $\models \alpha_1 \leftrightarrow \alpha_2$, then $\models (\kappa_1 \circ \alpha_1) \leftrightarrow (\kappa_2 \circ \alpha_2)$
**(R5)** $\models ((\kappa \circ \alpha) \wedge \beta) \rightarrow (\kappa \circ (\alpha \wedge \beta))$
**(R6)** If $\not\models \neg(\kappa \circ \alpha) \wedge \beta$, then $\models (\kappa \circ (\alpha \wedge \beta)) \rightarrow ((\kappa \circ \alpha) \wedge \beta)$

A ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ is defined as *$\kappa$-faithful* if and only if $\min_{\prec} \mathcal{V} = Mod_{\mathcal{V}}(\kappa)$. A revision operator $\circ_{\mathscr{R}}$ (on $\mathcal{L}$) is *defined by a $\kappa$-faithful ranked model $\mathscr{R}$* if and only if $Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha) = \min_{\prec} Mod_{\mathcal{V}}(\alpha)$. Katsuno and Mendelzon [12] proved that for $\mathcal{V} = \mathscr{U}$, (*i*) every revision operator $\circ_{\mathscr{R}}$ defined by a $\kappa$-faithful ranked model $\mathscr{R}$ is an AGM revision operator (on $\mathcal{L}$), and (*ii*) for every AGM revision operator $\circ$ (on $\mathcal{L}$) for $\kappa$, there is a $\kappa$-faithful ranked model $\mathscr{R}$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha)$.

We show that $\circ$ can be expressed in $\overline{\mathcal{L}}$ using typicality. The key insight is to identify the knowledge base $\kappa$ to be revised with the formula $\overline{\top}$, while $\kappa \circ \alpha$ is identified with $\overline{\alpha}$.

**Proposition 4.** *Let $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ be any $\kappa$-faithful ranked model (with $\kappa \in \mathcal{L}$). Then $[\![\kappa \circ_{\mathscr{R}} \alpha]\!] = [\![\overline{\alpha}]\!]$ (for every $\alpha \in \mathcal{L}$). Conversely, let $\circ$ be any AGM revision operator (on $\mathcal{L}$) for $\kappa$. Then there is a $\kappa$-faithful ranked model $\mathscr{R}$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = [\![\overline{\alpha}]\!]$.*

This result shows that propositional AGM revision can be embedded in $\overline{\mathcal{L}}$. But we can take this a step further and extend revision to apply to the language $\overline{\mathcal{L}}$ as well. So, with $\mathcal{V}$ still fixed, we let $\mathcal{R}_{\mathcal{V}} = \{\mathscr{R} \mid \mathscr{R} = \langle \mathcal{V}, \prec \rangle\}$ and we fix a $\kappa \in \overline{\mathcal{L}}$ such that $\mathscr{R} \not\Vdash \neg\kappa$ for some $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}$. The definition of a revision operator $\circ$ is then the same as above, except that it is now with respect to $\overline{\mathcal{L}}$. And the definition of an AGM revision operator on $\overline{\mathcal{L}}$ is then one which satisfies (R1)–(R6), but with validity in the postulates understood to be modulo $\mathcal{R}_{\mathcal{V}}$ (that is, for $\alpha \in \overline{\mathcal{L}}$, $\models \alpha$ if and only if $\mathscr{R} \Vdash \alpha$ for every $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}$.) This gives us a representation result similar to that of Katsuno and Mendelzon, but with the revision operator defined on $\overline{\mathcal{L}}$.

**Theorem 2.** *Let $\mathscr{R}$ be a $\kappa$-faithful ranked model. Then $\circ_{\mathscr{R}}$ is an AGM revision operator on $\overline{\mathcal{L}}$ for $\kappa$. Conversely, for every AGM revision operator $\circ$ on $\overline{\mathcal{L}}$ for $\kappa$ there is a $\kappa$-faithful ranked model $\mathscr{R}$ such that $Mod_{\mathcal{V}}(\kappa \circ \alpha) = Mod_{\mathcal{V}}(\kappa \circ_{\mathscr{R}} \alpha)$.*

## 6 Rational Consequence on $\overline{\mathcal{L}}$

We have seen in Section 5 that typicality can be used to express propositional AGM belief revision, as well as AGM belief revision defined for PTL. From Proposition 2 we know that rational consequence for propositional logic can be expressed in PTL. In this section we complete the picture by showing that (*i*) rational consequence for PTL can be expressed in PTL itself, a result analogous to Theorem 2, and (*ii*) that the expected connection between rational consequence and AGM revision for PTL does indeed hold.

As in Section 5, we start by fixing a set $\mathcal{V} \subseteq \mathscr{U}$. In this case, however, $\mathcal{V}$ is allowed to be empty as well. Then we let $\mid\!\sim$ be a binary relation on $\overline{\mathcal{L}}$. We say that $\mid\!\sim$ is a *rational consequence relation on* $\overline{\mathcal{L}}$ if and only if it satisfies the seven rationality properties from Section 2. In this case (as in Section 5) $\models$ is understood to be validity modulo $\mathcal{R}_{\mathcal{V}}$: $\models \alpha$ if and only if for every $\mathscr{R} \in \mathcal{R}_{\mathcal{V}}$, $\mathscr{R} \Vdash \alpha$. As was done in Section 2, given a ranked model $\mathscr{R}$, a pair $(\alpha, \beta)$ is in the consequence relation defined by $\mathscr{R}$ (denoted as $\alpha \mid\!\sim_{\mathscr{R}} \beta$) if and only if $\min_{\prec}[\![\alpha]\!] \subseteq [\![\beta]\!]$. In this case, however, $\alpha$ and $\beta$ are taken to be elements of $\overline{\mathcal{L}}$ and not just of $\mathcal{L}$. From this we get the following:

**Theorem 3.** *Every $\mid\!\sim_{\mathscr{R}}$ defined by some $\mathscr{R}$ is a rational consequence relation on $\overline{\mathcal{L}}$. Conversely, for every rational consequence relation $\mid\!\sim$ on $\overline{\mathcal{L}}$ there exists a ranked model $\mathscr{R}$ such that $\mid\!\sim_{\mathscr{R}} = \mid\!\sim$.*

It is worth mentioning that the proof of Theorem 3 makes use of Theorem 2, as well as the connection between AGM revision and rational consequence for PTL in the style of Gärdenfors and Makinson [9], which we now proceed to describe. First we consider the following additional property on defeasible consequence relations:

$$\text{(Cons)}\ \ \top \not\mid\!\sim \bot$$

It is easy to see that for a ranked model $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$, $\top \mid\!\sim_{\mathscr{R}} \bot$ if and only if $\mathcal{V} = \emptyset$. By insisting that (Cons) holds, we are restricting ourselves to ranked models in which $\mathcal{V} \neq \emptyset$, a restriction that is necessary to comply with postulate (R3) for AGM belief revision. So, we consider only the case where the (fixed) set $\mathcal{V}$ is non-empty.[4]

Intuitively, given a rational consequence relation $\mid\!\sim$ and a belief revision operator $\circ$ for a knowledge base $\kappa$, the idea is to (*i*) associate $\kappa$ with all $\beta$s such that $\top \mid\!\sim \beta$ holds and (*ii*) to associate the consequences of $\kappa \circ \alpha$ with all the $\beta$s such that $\alpha \mid\!\sim \beta$ holds.

For a rational $\mid\!\sim$ on $\overline{\mathcal{L}}$, let $C^{\mid\sim} = \{\alpha \in \overline{\mathcal{L}} \mid \top \mid\!\sim \alpha\}$ and let $\mathcal{K}^{\mid\sim}$ be the set of logically strongest formulas (modulo $\mathcal{R}_{\mathcal{V}}$) to be defeasibly concluded from $\top$. That is,

$$\mathcal{K}^{\mid\sim} = \{\alpha \in C^{\mid\sim} \mid \text{ for all } \beta \in C^{\mid\sim}, \text{ if } \models \beta \to \alpha, \text{ then } \models \alpha \to \beta\}.$$[5]

**Theorem 4.** *Let $\mid\!\sim$ be a rational consequence relation on $\overline{\mathcal{L}}$ also satisfying (Cons), and let $\kappa \in \mathcal{K}^{\mid\sim}$. There is an AGM revision operator $\circ$ on $\overline{\mathcal{L}}$ for $\kappa$ such that $\alpha \mid\!\sim \beta$ if and only if $\models (\kappa \circ \alpha) \to \beta$. Conversely, let $\kappa$ be any element of $\overline{\mathcal{L}}$ such that $\not\models \neg\kappa$, and let $\circ$ be an AGM revision operator on $\overline{\mathcal{L}}$ for $\kappa$. Then there is a rational consequence relation $\mid\!\sim$ on $\overline{\mathcal{L}}$ also satisfying (Cons) such that $\alpha \mid\!\sim \beta$ if and only if $\models (\kappa \circ \alpha) \to \beta$.*

[4] It is easy to see that if $\mathcal{V} = \emptyset$, then $\overline{\mathcal{L}} \times \overline{\mathcal{L}}$ is the only rational consequence relation satisfying all seven rationality properties, that $\mathscr{R} = \langle \emptyset, \emptyset \rangle$ is the only ranked model, and that $\mid\!\sim_{\mathscr{R}} = \overline{\mathcal{L}} \times \overline{\mathcal{L}}$.

[5] Where $\models$ is understood to mean validity modulo $\mathcal{R}_{\mathcal{V}}$.

## 7 Entailment for PTL

In this section we focus on what is perhaps the central question concerning PTL from the perspective of knowledge representation and reasoning: What does it mean for a PTL formula to be *entailed* by a (finite) knowledge base $\mathcal{K}$? Formally, we view an entailment relation as a binary relation $\models^*$ from the power set of the language under consideration (in this case $\overline{\mathcal{L}}$) to the language itself. Its associated *consequence* relation is defined as: $\mathit{Cn}^*(\mathcal{K}) = \{\alpha \mid \mathcal{K} \models^* \alpha\}$. Before looking at specific candidates, we propose some desired properties for such an entailment relation. The obvious place to start is to consider the properties for Tarskian consequence below.

**(Inclusion)** $\mathcal{K} \subseteq \mathit{Cn}^*(\mathcal{K})$
**(Idempotency)** $\mathit{Cn}^*(\mathcal{K}) = \mathit{Cn}^*(\mathit{Cn}^*(\mathcal{K}))$
**(Monotonicity)** If $\mathcal{K}_1 \subseteq \mathcal{K}_2$, then $\mathit{Cn}^*(\mathcal{K}_1) \subseteq \mathit{Cn}^*(\mathcal{K}_2)$

Inclusion and Idempotency are both properties we want to have satisfied, but Monotonicity is not. To see why not, it is enough to refer to the classic example in nonmonotonic reasoning: Let $\mathcal{K}_1 = \{p \to b, \overline{b} \to f\}$ (penguins are birds, and birds typically fly), and let $\mathcal{K}_2 = \mathcal{K}_1 \cup \{\overline{p} \to \neg f\}$ (add to $\mathcal{K}_1$ that penguins typically do not fly). We want $\overline{p} \to f \in \mathit{Cn}^*(\mathcal{K}_1)$ (penguins typically fly as a consequence of $\mathcal{K}_1$), but we want $\overline{p} \to f \notin \mathit{Cn}^*(\mathcal{K}_2)$ (penguins typically fly *not* as a consequence of $\mathcal{K}_2$), thereby invalidating Monotonicity.

In addition to Inclusion and Idempotency we require $\models^*$ to behave classically when presented with propositional information only (below $\models$ denotes classical entailment):

**(Classic)** If $\mathcal{K} \subseteq \mathcal{L}$, then for every $\alpha \in \mathcal{L}$, $\mathcal{K} \models^* \alpha$ iff $\mathcal{K} \models \alpha$

Therefore, we also require that the classical consequences of a knowledge base expressed in $\overline{\mathcal{L}}$ be classically closed (below $\mathit{Cn}(\cdot)$ refers to classical consequence of $\mathcal{L}$):

**(Classic Closure)** $\mathit{Cn}^*(\mathcal{K}) \cap \mathcal{L} = \mathit{Cn}(\mathit{Cn}^*(\mathcal{K}) \cap \mathcal{L})$

We now consider an obvious candidate for entailment: the standard Tarskian notion of entailment applied to the semantics of PTL:

$\mathcal{K} \models^T \alpha$ iff every ranked model $\mathscr{R}$ satisfying $\mathcal{K}$ also satisfies $\alpha$

It is easy to show that $\models^T$ satisfies Inclusion, Idempotency, Classic, and Classic Closure. However, it also satisfies Monotonicity, which eliminates it from contention as a viable form of entailment. Moreover, there is an additional argument against the use of $\models^T$ as well, one that is based on an adaptation of a result obtained by Lehmann and Magidor in the propositional case [14]. To make the argument, we first present a result showing that all formulas of $\overline{\mathcal{L}}$ can be rewritten as statements of rational consequence:

**Lemma 1.** *For every $\mathscr{R}$ and $\alpha \in \overline{\mathcal{L}}$, $\mathscr{R} \Vdash \alpha$ if and only if $\mathscr{R} \Vdash \overline{\neg\alpha} \to \bot$ if and only if $\neg\alpha \mid\!\sim_{\mathscr{R}} \bot$. Conversely for every $\mathscr{R}$ and $\alpha, \beta \in \overline{\mathcal{L}}$, $\alpha \mid\!\sim_{\mathscr{R}} \beta$ if and only if $\mathscr{R} \Vdash \overline{\alpha} \to \beta$.*

We can therefore think of $\overline{\mathcal{L}}$ as a language for expressing defeasible consequence on $\overline{\mathcal{L}}$ with $\mid\!\sim$ viewed as the only main connective. More precisely, let $\overline{\mathcal{L}}_{\mid\sim} = \{\alpha \mid\!\sim \beta \mid \alpha, \beta \in \overline{\mathcal{L}}\}$, and for any ranked model $\mathscr{R}$, let $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$ if and only if $\alpha \mid\!\sim_{\mathscr{R}} \beta$. The next result shows that the languages $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}_{\mid\sim}$ are equally expressive.

**Proposition 5.** *For every $\mathscr{R}$ and $\alpha \mid\!\sim \beta \in \overline{\mathcal{L}}_{\mid\sim}$, $\mathscr{R} \Vdash \alpha \mid\!\sim \beta$ if and only if $\mathscr{R} \Vdash \overline{\alpha} \to \beta$. Conversely, for every $\mathscr{R}$ and $\alpha \in \overline{\mathcal{L}}$, $\mathscr{R} \Vdash \alpha$ if and only if $\mathscr{R} \Vdash \neg\alpha \mid\!\sim \bot$.*

$\overline{\mathcal{L}}_{\mid\sim}$ is similar to the language for conditional knowledge bases studied by Lehmann and Magidor, but with the propositional component replaced by $\overline{\mathcal{L}}$ (i.e., $\mid\!\sim \subseteq \overline{\mathcal{L}} \times \overline{\mathcal{L}}$).

Based on this we restate entailment in terms of the language $\overline{\mathcal{L}}_{\mid\sim}$, and propose an additional property that any appropriate notion of entailment should satisfy. Let $\mathcal{K}$ be a (finite) subset of $\overline{\mathcal{L}}_{\mid\sim}$, let $\models^*$ be a (potential) entailment relation from $\mathscr{P}(\overline{\mathcal{L}}_{\mid\sim})$ to $\overline{\mathcal{L}}_{\mid\sim}$, and let $\mid\!\sim^*_{\mathcal{K}}$ be a defeasible consequence relation on $\overline{\mathcal{L}}$ obtained from $\models^*$ as follows: $\alpha \mid\!\sim^*_{\mathcal{K}} \beta$ if and only if $\mathcal{K} \models^* \alpha \mid\!\sim \beta$.

**(Rationality)** For every finite $\mathcal{K} \subseteq \overline{\mathcal{L}}_{\mid\sim}$, the consequence relation $\mid\!\sim^*_{\mathcal{K}}$ obtained from $\models^*$ should be a *rational*

Rationality is essentially the property for the entailment of propositional conditional knowledge bases proposed by Lehmann and Magidor [14], but applied to $\overline{\mathcal{L}}_{\mid\sim}$. Based on their results, it follows that $\models^T$ (defined on $\overline{\mathcal{L}}_{\mid\sim}$) does not satisfy Rationality. In fact, analogous to one of their results, we have the following result.

**Proposition 6.** *For finite $\mathcal{K} \subseteq \overline{\mathcal{L}}_{\mid\sim}$, let $\mid\!\sim^{\mathcal{K}} = \{(\alpha, \beta) \mid \alpha \mid\!\sim \beta \in \mathcal{K}\}$, and let $\mid\!\sim^P$ be the intersection of all preferential consequence relations on $\overline{\mathcal{L}}$ containing $\mid\!\sim^{\mathcal{K}}$. For the consequence relation $\mid\!\sim^T_{\mathcal{K}}$ obtained from $\models^T$, it follows that $\mid\!\sim^*_{\mathcal{K}} = \mid\!\sim^P$ is a preferential consequence relation, but not necessarily a rational consequence relation.*

Since $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}_{\mid\sim}$ are equally expressive, Proposition 6 provides additional evidence that $\models^T$ is not an appropriate form of entailment.

### 7.1 Rational Closure for PTL

Having shown that $\models^T$ is *not* an appropriate form of entailment for PTL, we now turn our attention to a proposal for an appropriate version of entailment. It is the notion of the *rational closure* of a conditional knowledge base, proposed by Lehmann and Magidor for the propositional case, applied to $\overline{\mathcal{L}}_{\mid\sim}$.

**Definition 5.** *Let $\mid\!\sim_0$ and $\mid\!\sim_1$ be rational consequence relations. $\mid\!\sim_0$ is preferable to $\mid\!\sim_1$ (written $\mid\!\sim_0 \ll \mid\!\sim_1$) if and only if*

- *there is an $\alpha \mid\!\sim \beta \in \mid\!\sim_1 \setminus \mid\!\sim_0$ s.t. for all $\gamma$ s.t. $\gamma \vee \alpha \mid\!\sim_0 \neg\alpha$ and for all $\delta$ s.t. $\gamma \mid\!\sim_0 \delta$, we also have $\gamma \mid\!\sim_1 \delta$;*
- *for every $\gamma, \delta \in \mathcal{L}$, if $\gamma \mid\!\sim \delta$ is in $\mid\!\sim_0 \setminus \mid\!\sim_1$, then there is an assertion $\rho \mid\!\sim \nu$ in $\mid\!\sim_1 \setminus \mid\!\sim_0$ s.t. $\rho \vee \gamma \mid\!\sim_1 \neg\gamma$.*

The motivation for $\ll$ here is essentially that for the same ordering for the propositional case provided by Lehmann and Magidor [14]. Given $\mathcal{K} \subseteq \overline{\mathcal{L}}$, the idea is now to define the rational closure as the most preferred (with respect to $\ll$) of all those rational consequence relations which include $\mathcal{K}$.

**Lemma 2.** *Let $\mathcal{K}$ be a finite subset of $\overline{\mathcal{L}}_{\mid\sim}$ and let $\mid\sim^{\mathcal{K}} = \{(\alpha, \beta) \mid \alpha \mid\sim \beta \in \mathcal{K}\}$. There is a unique rational consequence relation containing $\mid\sim^{\mathcal{K}}$ which is preferable (with respect to $\ll$) to all other rational consequence relations containing $\mid\sim^{\mathcal{K}}$.*

This allows us to define the rational closure $\models^{rc}$ of a knowledge base on $\overline{\mathcal{L}}_{\mid\sim}$.

**Definition 6.** *For finite $\mathcal{K} \subseteq \overline{\mathcal{L}}_{\mid\sim}$, let $\mid\sim^{\mathcal{K}} = \{(\alpha, \beta) \mid \alpha \mid\sim \beta \in \mathcal{K}\}$, and let $\mid\sim^{rc}$ be the (unique) rational consequence relation containing $\mid\sim^{\mathcal{K}}$ which is preferable (with respect to $\ll$) to all other rational consequence relations containing $\mid\sim^{\mathcal{K}}$. Then $\alpha \mid\sim \beta$ is in the rational closure of $\mathcal{K}$ (written as $\mathcal{K} \models^{rc} \alpha \mid\sim \beta$) if and only if $\alpha \mid\sim^{rc} \beta$.*

Definition 6 gives us a notion of rational closure for $\overline{\mathcal{L}}_{\mid\sim}$. Since $\overline{\mathcal{L}}$ and $\overline{\mathcal{L}}_{\mid\sim}$ are equally expressive, we can use Definition 6 to define rational closure for $\overline{\mathcal{L}}$ as well:

**Definition 7.** *Let $\mathcal{K} \subseteq \overline{\mathcal{L}}$, $\alpha \in \overline{\mathcal{L}}$, and let $\mathcal{K}^{\mid\sim} = \{\neg\beta \mid\sim \bot \mid \beta \in \mathcal{K}\}$. $\alpha$ is in the rational closure of $\mathcal{K}$ (written as $\mathcal{K} \models^{rc} \alpha$) if and only if $\neg\alpha \mid\sim \bot$ is in the rational closure of $\mathcal{K}^{\mid\sim}$.*

It is not hard to show that rational closure satisfies Inclusion, Idempotency, Classic, Classic Closure, and Rationality, but not Monotonicity. It is therefore a reasonable candidate for entailment for PTL.

### 7.2 Minimum Entailment for PTL

In this section we turn our attention to another proposal for entailment for $\overline{\mathcal{L}}$ based on a semantic construction. It is inspired by a proposal by Giordano et al. [10]. The idea is to define a partial order on a certain subclass of ranked models satisfying a knowledge base $\mathcal{K} \subseteq \overline{\mathcal{L}}$, with models lower down in the ordering being viewed as more 'conservative', in the sense that one can draw fewer conclusions from them, and therefore being more preferred. For $\mathcal{K} \subseteq \overline{\mathcal{L}}$, let $\mathcal{V}^{\mathcal{K}}$ be the elements of $\mathscr{U}$ permitted by $\mathcal{K}$: $\mathcal{V}^{\mathcal{K}} = \{v \mid v \in \mathcal{V}$ for some $\mathscr{R} = \langle \mathcal{V}, \prec \rangle$ s.t. $\mathscr{R} \Vdash \mathcal{K}\}$. And let $\mathcal{R}^{\mathcal{K}} = \{\mathscr{R} = \langle \mathcal{V}^{\mathcal{K}}, \prec \rangle \mid \mathscr{R} \Vdash \mathcal{K}\}$. Now, for any $\mathscr{R} = \langle \mathcal{V}^{\mathcal{K}}, \prec \rangle \in \mathcal{R}^{\mathcal{K}}$, let $\mathcal{V}_0^{\mathscr{R}} = \min_{\prec} \mathcal{V}^{\mathcal{K}}$, and for $i > 0$ let $\mathcal{V}_i^{\mathscr{R}} = \min_{\prec} \left( \mathcal{V}^{\mathcal{K}} \setminus (\cup_{j=0}^{j=i-1} V_j^{\mathscr{R}}) \right)$. So $\mathcal{V}_0^{\mathscr{R}}$ contains the elements of $\mathcal{V}^{\mathcal{K}}$ lowest down w.r.t. $\prec$, $\mathcal{V}_1^{\mathscr{R}}$ contains the elements of $\mathcal{V}^{\mathcal{K}}$ just above $\mathcal{V}_0^{\mathscr{R}}$ w.r.t. $\prec$, etc. Next, for every $v \in \mathcal{V}^{\mathcal{K}}$ we define the *height* of $v$ in $\mathscr{R}$ as $h^{\mathscr{R}}(v) = i$ if and only if $v \in \mathcal{V}_i^{\mathscr{R}}$. And based on that, we define the partial order $\preceq$ on $\mathcal{R}^{\mathcal{K}}$ as follows: $\mathscr{R}_1 \preceq \mathscr{R}_2$ if and only if for every $v \in \mathcal{V}^{\mathcal{K}}$, $h^{\mathscr{R}_1}(v) \leq h^{\mathscr{R}_2}(v)$. From this we get:

**Proposition 7.** *For every $\mathcal{K} \subseteq \overline{\mathcal{L}}$, the partial order $\preceq$ on the elements of $\mathcal{R}^{\mathcal{K}}$ has a unique minimum element.*

This allows us to provide a definition for the *minimum entailment* of a knowledge base.

**Definition 8.** *Let $\mathcal{K} \in \overline{\mathcal{L}}$, $\alpha \in \overline{\mathcal{L}}$, and $\mathscr{R}^{\mathcal{K}}$ be the (unique) minimum element of $\mathcal{R}^{\mathcal{K}}$ w.r.t. the partial order $\preceq$ on $\mathcal{R}^{\mathcal{K}}$. Then $\alpha$ is in the minimum entailment of $\mathcal{K}$ ($\mathcal{K} \models^{\min} \alpha$) if and only if $\mathscr{R}^{\mathcal{K}} \Vdash \alpha$.*

It can be shown that minimum entailment satisfies Inclusion, Idempotency, Classic, Classic Closure, and Rationality, but not Monotonicity. As for rational closure, it is a reasonable candidate for entailment for $\overline{\mathcal{L}}$. In fact, the connection between rational closure and minimal entailment may even be closer than that. There is strong evidence to support the conjecture that they actually coincide.

## 8 Related Work

To the best of our knowledge, the first attempt to formalize a notion of typicality in the context of defeasible reasoning was that by Delgrande [8]. Given the relationship between our constructions and those by Kraus et al., most of the remarks in the comparison made by Lehmann and Magidor [14, Section 3.7] are applicable in comparing Delgrande's approach to ours and we do not repeat them here.

Crocco and Lamarre [7] as well as Boutilier [2] have also explored the links between defeasible consequence relations and notions of normality similar to ours. In particular, Boutilier showed that nonmonotonic consequence can be embedded in conditional logics via a binary modality $\Rightarrow$. Here we have considered a unary operator. The links between our $^{-}$ and the conditional $\Rightarrow$ remain to be explored, though. Our conjecture at the moment is that they provide for the same expressivity.

In a description logic setting, Giordano et al. [11] also study notions of typicality. Semantically, they do so by placing an (absolute) ordering on *objects* in first-order domains in order to define versions of defeasible subsumption relations in the description logic $\mathcal{ALC}$. The authors moreover extend the language of $\mathcal{ALC}$ with an explicit typicality operator $\mathbf{T}$ of which the intended meaning is to single out instances of a concept that are deemed as 'typical'. That is, given an $\mathcal{ALC}$ concept $C$, $\mathbf{T}(C)$ denotes the most typical individuals having the property of being $C$ in a particular DL interpretation.

Giordano et al.'s approach defines rational versions of the DL subsumption relation $\sqsubseteq$. Nevertheless, they do not provide representation results and do not address the question of entailment either. In a recent paper [5] we have addressed precisely these issues in DLs. Even though here we have investigated typicality in a propositional setting, we expect that our representation result and constructions for the rational closure (as well as the links with belief revision) can be lifted to the DL case.

Britz et al. [3] investigate another embedding of propositional preferential reasoning in modal logic. In their setting, the modular ordering is an accessibility relation on possible worlds, axiomatized via a modal operator $\Box$. Our typicality operator can be defined in terms of their modality as $\overline{\alpha} \equiv_{\text{def}} \Box\neg\alpha \wedge \alpha$. The modal sentence $\Box\neg\alpha \wedge \alpha$ says that the worlds satisfying it are $\alpha$-worlds and whatever world is more preferable than these is a $\neg\alpha$-world. In other words, these are the minimal $\alpha$-worlds. The general case of defining Britz et al.'s modality in terms of our typicality operator is not possible, but in a finitely generated language as we consider here, the logics become identical.

Britz and Varzinczak [6] investigate another aspect of defeasibility by introducing (non-standard) modal operators allowing us to talk about relative normality in accessible worlds. With their defeasible versions of modalities, it is possible to make statements of the form "$\alpha$ holds in all of the normal (typical) accessible worlds", thereby capturing defeasibility of what is 'expected' in target worlds. This allows for the definition of a family of modal logics in which defeasible modes of inference can be expressed, and which can be integrated with existing $\mid\!\sim$-based modal logics [4].

## 9 Concluding Remarks

The contributions of this work are as follows: (*i*) We present the logic PTL which provides a formal account of typicality with which to capture the most typical situations in which a given formula holds; (*ii*) We show that we can embed the (propositional) KLM framework within PTL, and we also define rational consequence on PTL itself; (*iii*) We establish a connection between rational consequence and belief revision, both on PTL, and (*iv*) We investigate appropriate notions of entailment for PTL and propose two candidates.

For future work we are interested in algorithms for computing the appropriate forms of entailment for PTL, specifically algorithms that can be reduced to validity checking for PTL. It follows indirectly from results by Lehmann and Magidor [14] that this type of entailment has the same worst-case complexity of validity checking for PTL. Given the links with modal logic, we know that this is at least in PSPACE. Finally we plan to extend PTL to more expressive logics such as description logics and modal logics.

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